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Olaf Pedersen

A Survey of the *Almagest*

With Annotation and New Commentary by Alexander Jones

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Ptolemy. Sculpture by Jörg Syrlin the Elder in Ulm Cathedral; about 1469–74.

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ISBN 978-0-387-84825-9 e-ISBN 978-0-387-84826-6
DOI 10.1007/978-0-387-84826-6
Springer New York Dordrecht Heidelberg London

Mathematics Subject Classification (2010): 85-03

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Foreword to the Revised Edition

There are many reasons why one should wish to read Ptolemy's *Almagest*. As well as being a masterpiece of scientific writing, it was the one major work on the scientific modelling of the celestial phenomena that survived from Greco-Roman civilization. It is our chief informant on Greek mathematical astronomy during the interval of its highest development (from the second century B.C. to about A.D. 150, roughly the date when the *Almagest* was published), and its influence shaped the astronomy of later antiquity, medieval Islam, and early modern Europe.

But it was never an easy read. As Ptolemy says in his preface, he wrote concisely and counted on his reader to be already rather experienced in the subject: comfortable with the deductive geometry of Euclid's *Elements*, proficient in numerical calculation involving a place-value system for fractions, and familiar with the visible phenomena of the heavenly bodies and the way they were recorded in observation reports. In Ptolemy's own day there were probably very few readers who could appreciate the treatise as much more than a collection of tables useful for astrological calculations, embedded in interminable stretches of indigestible mathematical prose. By the fourth century A.D., two hundred years after Ptolemy, the *Almagest* had been adopted as a school text for the most advanced philosophical and mathematical students in Alexandria, and its teachers Pappus and Theon felt the need to help their pupils along by composing commentaries, longer than the *Almagest* itself, that filled out Ptolemy's terse logic into a more leisurely, nay tedious, exposition. The modern student, who is more remote from Ptolemy's world and its scientific and mathematical conventions, needs help with a wider range of topics, but no longer expects to be led by the hand through each step of each of Ptolemy's mathematical arguments; and judicious use of mathematical notation helps one to grasp what Ptolemy expresses verbally in sentences that can run to several lines, and relate it to the mathematical tools we all learn in school. Where the pupils of Pappus and Theon read the *Almagest* to master its subject, we read it as a historical document representing an astronomy whose subject matter lies at the extreme margins of modern astronomy and astrophysics. We therefore have to learn the basic facts about naked-eye astronomy that today's schools no longer teach, and we look for information about the book's intellectual background and legacy.

Olaf Pedersen's *Survey of the Almagest* follows in a long tradition of introductory companions, part paraphrase and part commentary, that can be traced back through Paul Tannery (*Recherches sur l'histoire de l'astronomie ancienne*, 1893) and J. B. Delambre (the second volume of his *Histoire de l'astronomie ancienne*, 1817) to Regiomontanus' *Epitome of Ptolemy's Almagest* (printed posthumously in 1496) and

indeed to Proclus' excellent *Sketch of the Astronomical Models* (mid fifth century). The eighty years since the last "survey of the *Almagest*" had seen considerable advances in our knowledge of the ancient astronomy, and of course there had also been great changes in the way students learn and what background knowledge they could be expected to have. Pedersen's *Survey* earned the universal praise of historians of early astronomy, and since 1974 it has been the first book one puts in the hands of a student approaching the *Almagest*.

Olaf Pedersen (1920–1997) was trained as a theoretical physicist before turning to the history of science in his doctoral study.[1] His earlier work centered on medieval European physics and astronomy, and his interest in Ptolemy came by way of research on medieval texts known as *Theorica* that present detailed descriptions of versions of the Ptolemaic system of celestial models. The *Survey* arose out of lectures on the *Almagest* that he gave in the Department of History of Science that he founded at the University of Aarhus, and its effectiveness as an introduction to a singularly difficult book reflects this pedagogical origin.

The English-reading student of the *Almagest* in 2010 has a substantial advantage over his or her predecessor of 1974 in being able to read Ptolemy's text in the splendid translation by G. J. Toomer (1984). It is very easy to find the passages in Toomer's translation corresponding to Pedersen's discussions, since Pedersen's references give the volume and page number in Heiberg's standard edition of the Greek text, and these also appear in Toomer's margins. Toomer's footnotes, appendix of worked examples of computations, and index are also invaluable as a supplement to Pedersen. "Book I" of O. Neugebauer's *A History of Ancient Mathematical Astronomy*, which was published one year after Pedersen's *Survey*, provides a detailed analysis of the mathematical astronomy in the *Almagest* that is often complementary to Pedersen's.

In this revised edition, Pedersen's text has been left unaltered except for the correction of typographic and other isolated errors that could be corrected in-line.[2] Where scholarship of which Pedersen was unaware or that has appeared since 1974 calls for modification or expansion of his discussion, a reference in the margin alerts the reader to a supplementary note at the end of the volume. This supplement, like the supplementary bibliography following it, is selective and does not pretend to supply a comprehensive review of the recent scholarly literature.

Alexander Jones

1) For a biography see North (1998).
2) For many of these, as well as other corrections in the supplementary notes, I am indebted to the reviews by G. J. Toomer (1977), G. Saliba (1975), and V. E. Thoren (1977).

Contents

FOR
THE MASTER AND FELLOWS
OF ST. EDMUND'S HOUSE
CAMBRIDGE

Preface

This book contains a survey of the *Almagest*. It is the outcome of a course of lectures given on several occasions, and its only aim is to help students of the history of astronomy to understand and appreciate Ptolemy's great and classical work. Therefore the emphasis is on the various astronomical theories and their structural relationships, while a critical analysis of Ptolemy's observational efforts falls outside the scope of the present study. Appendix A gives a list of all the dated observations quoted in the Almagest; it is compiled only for purposes of reference and is not based on any re-examination of Ptolemy's empirical data.

The problem of notation has been rather difficult and it has been necessary to go beyond the means that Ptolemy had at his disposal. On the one hand, many essential features of the methods and models employed by Ptolemy become both clearer and more accessible to modern readers if common mathematical expressions are introduced as condensed, formalized versions of procedures explained by Ptolemy in the form of verbal or numerical statements. On the other hand, Ptolemy's surprisingly non-technical and sometimes ambiguous terminology can be made more specific by means of concepts and terms borrowed from the vocabulary of Mediaeval Latin astronomers. I hope that these departures from the form and style of the Almagest will help to make the exposition useful to students whose principal interests are in the astronomy of the Middle Ages, or even in Copernicus and his immediate successors.

In establishing the mathematical formalism I must confess to a certain number of repetitions of equations and procedures. They are introduced in order to make it easier to study the theory of the motion of a particular kind of heavenly body, the various chapters on the motion of the Sun, the Moon, the superior, and the inferior planets being presented as more or less self-contained parts of the text.

No complete and detailed analysis of the Almagest has appeared since 1817 when Delambre published the second volume of his *Histoire de l'Astronomie Ancienne*. Since then a great number of papers and dissertations have dealt with particular problems in Ptolemaic astronomy. Most of these publications are listed in the bibliography and references to them are scattered throughout the text. Because of its limited purpose this survey does not pretend to have utilized all this material to its full extent. A few particular questions have been referred to the notes to each chapter.

A work like this cannot come to light without help and support from many different quarters. I am gratefully obliged to the Danish Council for Scientific Research which

supported the studies preparatory to this book and also facilitated its publication. My sincere thanks are due to Professor Mogens Pihl for recommending another book of mine for publication in this series, and for friendly and unfailing help and advice over many years. Likewise I wish to thank Mr. Torkil Olsen of the Odense University Press, and the Cooperative Printing Company, Odense, for their excellent services in producing a most difficult book.

This study was begun and finished in the hospitable and scholarly atmosphere of St. Edmund's House, Cambridge. My most heartfelt gratitude is due to the Master and Fellows of this college for making me one of their own and providing me with excellent working facilities. In the same vein I am grateful to my wife for her constant encouragement and great patience with a husband absorbed in burning midnight oil for years on end. I am greatly indebted to my friends and colleagues Dr. Michael A. Hoskin, Cambridge, Professor Willy Hartner, Frankfurt a.M., and Professor Owen Gingerich, Harvard, for reading parts of the manuscript and recommending its publication. The proofs were read by Professor Willy Hartner, Frankfurt a.M. and by Dr. John D. North, Oxford, who both made criticisms and suggestions for which I am most grateful. I am under a special obligation to my friend Dr. A. G. Drachmann, Copenhagen, who for several years put his copy of Heiberg's editon of Ptolemy's works at my disposal. The staff of the Institute for History of Science, Aarhus University, has been most helpful. In particular I wish to thank Dr. K. P. Moesgaard for his careful examination of the complete text, and Mr. Kurt Møller Pedersen, M. Sc., for his help with many numerical calculations. Dr. G. A. Dirac, Aarhus University, took upon himself the task of revising the text from a philological point of view; I am truly grateful for his assistance without which I should not have ventured to publish this book. I thank also Mrs. Tove Asmussen for help with some of the illustrations. Last, but not least, I wish to express my profound gratitude to my secretary, Mrs. Kate Larsen, whose untiring efforts have been the *causa efficiens* of the whole work at all its different stages.

O.P.

CHAPTER 1

The Almagest through the Ages

The *Almagest* has shared the fate of many other major works in the history of science. It has been talked about by many, but studied seriously only by the few. Yet it was just as important to ancient science as Newton's *Principia* was to the 17th century, and there is no question that it was a greater scientific achievement than the *De revolutionibus* which has obliterated its fame, just as Copernicus has outshone Ptolemy as an astronomical genius. The Almagest was the culmination of Greek astronomy, and unrivalled in Antiquity as an example of how a large and important class of natural phenomena could be described in mathematical terms in such a way that their future course could be predicted with reasonable precision. It taught scientists of many ages how geometrical and kinematical models could be constructed and, by means of empirical data stemming from careful observations, made to simulate nature in a way which came to influence scientific method until the present day. It is true that the Babylonians had succeeded in developing highly sophisticated, algebraic methods to deal with the phenomena. But, as far as we know, they never tried to summarize either their methods or their results in a comprehensive work comparable with this brilliant exposition of everything achieved by Ptolemy himself and by the most remarkable of his predecessors among the Greek astronomers.

The Life and Work of Ptolemy

We know but little about the author of the Almagest. His life and personality are remarkably obscure compared with what we know of some other eminent scientists of the ancient world, but the reason is close at hand. Thus Plutarch gave us vivid biographical sketches of both Aristotle and Archimedes – but only because the former happened to be the teacher of a great warrior, and because the latter was killed by a soldier of a general in whom Plutarch was interested. Now Plutarch must be excused for not writing the life of Ptolemy who was his contemporary. But the 3rd century lexicographer Diogenes Laertius does not even mention his name, and also later tradition concerning Ptolemy is very scarce. Everything points to the conclusion that he lived and died peacefully, unnoticed by the mighty of this world and devoting himself to the composition of scientific works of a highly technical nature, unable to attract the attention of the literati of late Antiquity. What little remains from their hands has been collected by Boll (1894) and Fischer (1932), together with a few

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_1,

fragments from later Arabic authors and of a more or less legendary nature – e.g. that he was of moderate height, with a pale face, a black, pointed beard, a small mouth and a mild and pleasant voice, etc. .... This is, in fact a standard description of a Stoic philosopher, without any historical value at all. The result is that our image of Ptolemy lies hidden in his own works from which at least some biographical information can be extracted[1]).

Let us first consider the list of the observations recorded in the Almagest (see Appendix A). About one third of the total number are due to Ptolemy himself, the first being an observation of an eclipse of the Moon made in Alexandria in the 9th year of Hadrian's reign, or more precisely A.D. 125 April 5. The last is an observation of the maximum elongation of Mercury made in the 4th year of Antoninus Pius, i.e. A.D. 141 February 2, also in Alexandria. This shows that Ptolemy collected the observational data for his planetary theories in the period A.D. 125–141, and that the Almagest must have been finished after the later date. But there is sufficient evidence to show that Ptolemy continued his scientific activity after he completed his Almagest. Thus we know that in the 10th year of Antoninus (A.D. 147/48) he erected a stele in the town of Canopus some 15 miles East of Alexandria. It was provided with an inscription giving improved parameters of the planetary models. Still better parameters are found in the so-called *Handy-Tables*, or *Tabulae manuales* which accordingly seem to be of an even later date (Stahlmann, 1960, and v. d. Waerden, 1953, 1958). Also the work on the *Planetary Hypotheses* as well as the astrological *Tetrabiblos* are later than the Almagest, as stated explicitly in the texts. Finally it seems that Ptolemy's great treatise on *Geography* is later than the Handy Tables. If we remember that Ptolemy also left a considerable number of other works on optics, harmonics, mathematics and philosophy, we must realize that a large span of years was required to produce them. Everything considered it would seem that Ptolemy's scientific activity went on from about A.D. 125 to well into the reign of Marcus Aurelius (161–180). This agrees well with a statement in Olympiodorus that Ptolemy worked in Canopus for 40 years (v. d. Waerden, 1957, col. 1789). Probably the dates of his birth and death are about A.D. 100 and A.D. 165.

Where Ptolemy was born is not known. His real name Claudios is Greek, whereas his second name Ptolemaios could be an indication that he came from one of the various Egyptian towns named after the Ptolemaic kings, perhaps Ptolemaïs Hermeiu in Middle Egypt. Purely legendary is the Mediaeval tradition that he was of the old royal stock. Certain is only the fact that he performed his observations at Alexandria as stated in several places in the Almagest. Accordingly the new instruments he invented and constructed must have been placed there in some kind of observatory. But the undoubted tradition that he erected his stele at the neighbouring Canopus shows a certain connection with this locality. Perhaps he had his home in this smaller town, which offered better possibilities for a quiet life of study than the noisy capital of the Hellenistic world.

1) Cf. to the following B. L. van der Waerden (1957). This brilliant survey has superseded earlier expositions by Bunbury et al. (1926), Sarton (1927), and others.

Since the reign of king Ptolemy I Soter (d. 282 BC) Alexandria had been famous for its school where e.g. Euclid had been a scholar, Archimedes a student, and Eratosthenes a librarian. Its libraries were unsurpassed in the Ancient world and contained several hundred thousand writings of both Greek and Oriental provenance, together with catalogues of a considerable part of the collections (Parsons, 1952). The list of observations referred to above shows that Ptolemy often searched them for records of earlier observations. Here he had access not only to the results of Hipparchus, Aristarchus, and the first Alexandrian astronomer Timocharis, but also to 5th century observations made in Athens by Meton and Euctemon, and even to very ancient Babylonian records enabling him to make use of a Lunar eclipse observed in Babylon as early as 721 B.C. There is only one reference to a contemporary scientist, Theon of Smyrna, who gave some of his observations from A.D. 127–132 to Ptolemy, who may have been his friend or pupil [X, 1; Hei 2, 296]. It was Ptolemy's personal merit that he was more careful than most other Hellenistic authors in quoting his sources and acknowledging his predecessors; but it was the great Alexandrian library which enabled him to do so.

The richness of the library is part of the explanation of a characteristic feature of Alexandrian learning. Critical scholarship and bibliography were born here, and from the beginning there was a marked predilection among scholars for large and comprehensive expositions. The famous *Elements* of Euclid are a good example of a handbook containing most of the achievements of previous Greek mathematicians, arranged in a logical order and therefore easy to use in further research. Now Ptolemy is remembered chiefly as an astronomer; but a look at the titles of his many extant or lost works reveals him as a scientist of much more catholic interests. It is almost as if he had intended to compile a huge "Encyclopedia of Applied Mathematics" as a counterpart to already existing comprehensive expositions of pure mathematics.

It has to be admitted that Ptolemy succeeded in carrying this project out. Nevertheless, it would be wrong to regard him as a mere compiler of the results of others. Because he always acknowledged his debts to earlier scientists his own results stand out clearly, and all his works contain important personal contributions. His scientific spirit and love of research appear clearly from the fact that he did not rest upon what he had already achieved: once the Almagest was finished, by a rare and remarkable intellectual effort Ptolemy immediately began to improve both the theories of latitude and the parameters of the Lunar model, as shown by the Canopus-inscription and the Handy Tables. But that such restless urges to better his own results are most clearly seen in his astronomy is, perhaps, a testimony that with all his widespread interests Ptolemy was in his heart an astronomer, as tradition maintained.

The Spread of Ptolemaic Astronomy

For more than a century after Ptolemy's death Hellenistic scientists were silent about his work; but there is every reason to suppose that it was held in high esteem in the Alexandrian School where the mathematician Pappus (about A.D. 300) wrote the first

of the many commentaries to the Almagest. Only Book 5 and 6 have survived (ed. Rome, 1931). In the second half of the 4th century A.D. a new commentary was published by another Alexandrian mathematician named Theon; most of this work is extant. According to tradition also Theon's famous daughter Hypatia wrote about Ptolemy, before she was murdered by a mob of Christian fanatics in A.D. 415 in foreboding of the approaching downfall of the Alexandrian School (Sarton, 1927, p. 386). But even after that date the Byzantine mathematician and neo-Platonic philosopher Proclus (410–485) was able to study astronomy at Alexandria. Later he became head of the old Platonic Academy at Athens, where he wrote his *Hypotyposis* or introduction to the astronomy of Hipparchus and Ptolemy (ed. Manitius, 1909). This was the last Greek work on Ptolemaic astronomy before the closing of Plato's Academy in A.D. 529 marked the end of the culture of the ancient world.

Already before Hellenistic civilization had reached its final stage Greek science had begun to penetrate the cultures of the East. From the troubled schools of Alexandria, Antiochia and Athens Greek scholars emigrated to Mesopotamia and Persia, and even in India the impact of Greek astronomy was felt in the great Sanskrit *Siddhantas* from about A.D. 500 onwards (Neugebauer, 1956, cf. Sen, 1971). These astronomical manuals are based on planetary theories of Greek origin, containing both pre-Ptolemaic and Ptolemaic features. The details of this transmission are not completely known and the spread of Greek astronomy among Oriental people is largely a matter of future research. But up to now we have no evidence that either the Persians, or the Indians, ever possessed the Almagest in their own languages.

The Almagest among the Arabs

On the other hand it is probable that there existed an Almagest translation in the Syriac language, through which the Arabs acquired much of their first knowledge of Ancient culture when they began to cultivate science and philosophy in Baghdad under the great Khalif al-Manṣur (754–775). But it is certain that their first knowledge of astronomy was derived from one of the Indian Siddhantas translated into Arabic about A.D. 773 (Brockelmann, I, 248). The result was that the first important astronomical works of the Muslim world were strongly influenced by Hindu astronomy. That was the case with e.g. the large *zīj*, or collection of astronomical tables with rules for their use, written by the Persian scientist al-Khwārizmī (d. ca. A.D. 850) (ed. Suter, 1914, transl. Neugebauer, 1962). Where specific Indian concepts and methods occasionally turn up in later Western astronomy (Neugebauer and Schmidt, 1952) they can usually be traced back to a Latin translation by Adelard of Bath (about A.D. 1126) of an Arabic version of this work.

It is a curious fact that Ptolemy became known to the Arabs as an astrologer before they learned about his astronomical achievements. Actually, already before or about A.D. 800 there was an Arabic translation of the Tetrabiblos made by the physician al-Baṭrīq (Suter, 1900, p. 4), while the Almagest had to wait until a later date before

becoming accessible to Muslim scholars. Unfortunately the literary history of the Arabic Almagest is rather obscure and we do not know for certain when and by whom the first translation was made. One possible candidate is the Jewish Rabbi Sahl al-Ṭabarī who probably flourished about the beginning of the 9th century (Suter, 1900, p. 14). However, most scholars are inclined to doubt the existence of a translation of that early date. Another possibility is an anonymous scholar at the Bagdhad court who in A.D. 827 made a translation based on the Greek text. An Arabic MS of this version is still preserved in the University Library at Leiden.

About the same time another translation was made in Baghdad by al-Ḥajjāj ibn Yusuf, whose version appeared in 829–30 and is said to have been based on a Syriac text (Suter, 1900, p. 208). It was this translation which happened to give the work its current title. It was called *Kitāb al-mijisti*, *kitāb* meaning 'book' and *al-mijisti* being the Arabic term which later was rendered into Latin as Almagest. There is no certain explanation of the origin of this term. The original Greek title of the work was Μαθηματικῆς Συντάξεως βιβλια ιγ̄ or *The 13 Books of Mathematical Collections.* Later it may have been called Μεγάλη σύνταξις or *The Great Collection*. Now the superlative of μεγάλη is μεγίστη, and it may well be that the Arabs simply provided the latter term with the article *al*, thus creating the mixed form *al-megiste*, from which the Latin *Almagest* emerged (Brockelmann, Suppl. I, 363).

What became the final and most widely used Arabic version of the Almagest was a translation from the Greek made by Isḥāq ibn Ḥunain (Brockelmann, I, 227; Sarton, 1927, pp. 600 and 611) who died in Baghdad 910/11 and was the son and collaborator of one of the most gifted, critical and prolific of the Baghdad translators, a Nestorian named Ḥunain ibn Isḥāq (810–877). Isḥāq lacked specialized knowledge of astronomy, but his translation was revised by the astronomer Thābit ibn Qurra (827–901) who was a famous translator in his own right (Suter 1900, p. 34).

In the course of the 9th century the Almagest was thus rescued from oblivion by Arabic speaking scholars. The result was that Muslim astronomy soon turned away from the earlier Hindu influence and acquired a definite Ptolemaic character. This appears both from elementary introductions, such as brief compilations made by al-Farghānī in the middle of the 9th century[2]), and from great scientific expositions, like the famous zīj of al-Battānī (ca. 850–929) upon which so much of our knowledge of Arabic astronomy is founded (ed. Nallino 1899–1907). The Almagest itself gave rise to a large number of more or less revised versions (Steinschneider, 1892) among which one of the most important was a long paraphrase by the Moorish astronomer Jābir ibn Afflaḥ, named Geber by the Latins (Vernet, 1963). It dates from about A.D. 1140 and was rendered into Latin by Gerard of Cremona[3]). It is fundamentally

2) In a Latin version by Johannes Hispalensis (12th century) this little manual was much used by Western astronomers. There is a printed edition *Brevis ac perutilis Compilatio Alfragani astronomorum peritissimi, quod ad rudimenta Astronomica est opportunum*, Norimbergæ 1537. It was also known as *Liber 30 differentiarum.*

3) This version was printed with the title *Gebri Filii Affla Hispalensis De astronomia libri IX, in quibus Ptolemæum, alioqui doctissimum, emendauit* as an appendix to the *Instrumentum primi mobilis* by Petrus Apianus, Norimbergæ 1534.

Ptolemaic, but very critical of Ptolemy on a number of technical points of small astronomical importance.

Ptolemy in the Latin Middle Ages

We must now turn our attention to the Latin Middle Ages, where the 12th century marks the great divide in the history of astronomy. Before that time Ptolemy was known by name only among authors whose astronomy was ultimately derived from Pliny and other compilers, but not from the original Hellenistic tradition. In the famous encyclopedia by St. Isidore of Seville, about A.D. 631, he is even mentioned as *Ptolemaeus rex Alexandriae* and thus confused with the Ptolemaic kings of Egypt (*Lib. Etym.* iii, 26). Accordingly Mediaeval MSS often depict him with a royal crown, an iconographical tradition persisting well into the era of printed books[4]) although the legend had been exploded long before[5]). Not until the 12th century was the real Ptolemy made known to Latin astronomers through a long series of translations of his works (Carmody, 1956). Also here astrology preceded astronomy, one of the first translations being a Latin version of the Tetrabiblos made in A.D. 1138 by Plato of Tivoli. In A.D. 1143 followed the Planisphaerium in a translation by Herman of Carinthia (H. Dalmata) and in 1154 the Optics was translated by Eugenius of Sicily. All these versions were based on previous Arabic translations. What we know of the Latin Almagest is due mainly to the researches of Haskins (1924), who has established the existence of several little known and not widely used translations. Actually at least four different versions were made, either from the Greek or the Arabic.

1) A translation made about A.D. 1160 in Sicily directly from the Greek, by an anonymous translator. It was discovered in 1909 by Lockwood and Bjørnbo independently and is the earliest known Latin version of the Almagest. Four MSS are known, of which only Vat. Lat. 2056 is complete. It has a preface by the translator with the incipit: *Eam pingendi Gratias antiqui feruntur habuisse consuetudinem*, whereas the incipit of the Ptolemaic text itself is: *Valde bene qui proprie philosophati sunt, o Sire*. The small number of MSS shows that this version was but little used. Actually, it was soon superseded by

2) a translation made in A.D. 1175 from the Arabic by the most industrious of

4) See e.g. Wm. Cunningham: *The Cosmographicall Glasse*, London, 1559, as reproduced in R. T. Gunther: *Early Science in Cambridge*, Oxford, 1937, p. 150.

5) This was done by an 11th-century Moorish astronomer Alī ibn Ridwan, called Haly Abenrudian by the Latins (see Suter, 1900, p. 103 and Sarton, 1927, p. 729), who in the preface to his Arabic translation of the *Tetrabiblos* showed that the stellar positions in the Almagest referred to an epoch in Imperial times, the longitudes of the fixed stars quoted being much too large to correspond to the times of the Ptolemies (because of precession). This fine example of historical criticism based on astronomy was known to Nicole Oresme who in the 14th century translated Haly's book into French, cfr. the MS Paris Bibl. Nat. F. Franç. 1348, 2vb–3ra: *Et trouuons que cetay ptholomee, qui composa le livre de almagesti, nomme les lieux des estoilles selon la rectification delles et selon ce que il doivent estre mises ou temps emperieres des romains et fust moult grant apres les temps des roys de alexandrie. Pour quoy nous entendon que cest ptholomee ci le ludian ne fu pas un des roys de alexandrie.* – But this late insight had no effect on Mediaeval history of astronomy in general.

all 12th century translators, Gerard of Gremona, who lived and worked at Toledo. It has the incipit (of Book I): *Bonum scire fuit quod sapientibus non deviantibus.* This became the standard version of the Almagest during the following three or four centuries. Carmody (1956) notes the existence of at least 32 extant MSS, without saying whether they are all of them complete.

3) A third translation with the incipit: *Bonum quidem fecerunt illi qui perscruti sunt scientiam philosophie* is extant only in fragments. According to Haskins (1924 p. 106) it was made from the Arabic, probably in Spain sometime during the 13th century.

4) Finally, there exists a fourth version of the first four books of the Almagest with the incipit *Preclare fecerunt qui corrigentes scienciam philosophie, O Syre.* Only one copy is known and the text can have had no wide circulation. The translator is anonymous; he worked before A.D. 1300 but we do not know where.

It is probable that other complete or partial translations were made, but in the present state of Mediaeval studies it is impossible to arrive at even an approximate survey of the whole field. Not even a check list of the extant MSS has been published. Peters and Knobel (1915) list 21 Greek, 8 Latin, and 4 Arabic MSS. Zinner (1925) has 24 MSS from what he calls the "German Cultural Domain" including Italy, Belgium, Poland, and Czekoslovakia, besides Germany and Austria, but gives only very little bibliographical information without distinguishing between the various translations.

Accordingly we must acknowledge that the literary tradition of the Almagest during the Middle Ages is very much a matter of future research, of equal importance to our understanding of both Mediaeval and Renaissance astronomy. Here we can venture only upon a few tentative conclusions.

First, the number of MSS and their distribution over the various translations shows that Gerard of Cremona's version was the most widely used. This is confirmed by the astronomical bibliographies of the Middle Ages, among which the most comprehensive is the famous 13th-century *Speculum Astronomie*, probably written by St. Albert the Great[6]). It first mentions a book with the incipit *Sphaera celi* ascribed to a certain Nembroth gigas[7]), but only to discard it as of too little use and erroneous (*paruum proficui et falsitates nonnullæ*). According to the Speculum a good introduction to astronomy is the Almagest: *Sed quod de hac scientia utilius inuenitur, est liber Ptolemæi Pheludensis, qui dicitur Græce megasti, Arabice almagesti, Latine minor perfectus, qui sic incipit: Bonum fuit scire, etc. quod tamen in eo diligentiæ causa dictum est prolixe ....* The incipit quoted here reveals that the reference is to Gerard's translation.

Second, we can conclude that in spite of the high esteem in which the Almagest was

6) *B. Alberti Magni* [...] *Opera*, Tom. V, Lugduni, 1651, pp. 656 ff. For the problem of the authorship, see Thorndike, 1923, vol. 2, pp. 692–717, and Nallino, 1948, p. 317.

7) The mythical astronomer Nembroth (or Nimrod) appears in Latin astronomy already in a 9th-century treatise *De forma celi* to which Haskins (1924 p. 336 ff.) has drawn attention, with the further remark that "Astronomical tables under his name are known to have been current in Arabic" (p. 338). However, this statement is not borne out in E. S. Kennedy (1956) in which the name does not occur.

held by Mediaeval astronomers it was but rarely studied from cover to cover. The small number of extant manuscripts points to the conclusion that the majority of astronomers never possessed a copy nor even had access to one in a library. The reason is not difficult to guess. The Almagest is a highly technical work which still to-day presents many difficulties and obscurities for a modern reader. It must have been much more difficult to a Mediaeval scholar equipped with less astronomical and mathematical knowledge. We have to remember that e.g. the *Elements* of Euclid were translated only a short time before the Almagest, and that it must have been an enormous task to assimilate such long and demanding treatises.

Now astronomy was one of the seven liberal arts, which all through the Middle Ages formed the basic framework of higher education, both in the schools of the 12th and the universities of the 13th and later centuries. This means that any Mediaeval student had to follow a course of astronomy as a preparation for his degree of Master of Arts, before he was allowed to go on to more specialized studies of law, medicine, or theology. At this introductory level it was clearly impossible to make use of the Almagest as a student's textbook. It remained a technical work for the advanced and competent scholar, while the ordinary student had to be provided with some more easily digestible manual. From the literary point of view the history of Mediaeval astronomy is the history of how Ptolemaic astronomy was assimilated, taught and moulded in its particular Latin form without direct use of the Almagest itself.

We shall not here follow this process in any detail, but only indicate a few of the main roads along which it proceeded. At a high level it was possible to use an abbreviated version or a paraphrase of the Almagest. Of this genre we have already mentioned the *De astronomia* by Jābir ibn Afflaḥ, which gave a summary of the whole work. A widely used manual wrongly ascribed to the same author was the *Almagestum parvum* or the *Almagesti minoris libri VI*. It was accessible in a Latin translation by Gerard of Cremona (Incipit: *Omnium recte philosophantium*) and gave an introduction to the more mathematical sections of the Almagest.

Even works of this kind were too difficult for students who had just learned their spherical astronomy from a very elementary manual like the widely used *Tractatus de sphera* by Johannes de Sacrobosco (ed. Thorndike, 1949). This extremely popular work contained about a page and a half of planetary theory, and had to be supplemented by an introduction to planetary theory at about the same level. Several such introductions are known, but the most popular was a certain *Theorica planetarum*. This work has been ascribed to Gerard of Cremona as well as to the 13th century astrologer Gerard of Sabbioneta, but seems to be the work of an unknown author from the latter half of the 13th century (Pedersen, 1962). The text was not free from errors, the most fatal being a wrong determination of the stationary points. But it had the advantage of being brief, with clear definitions of the main concepts of the various planetary models of the Almagest, illustrating them with a number of good diagrams. These obvious qualities ensured its success as the standard university textbook of planetary astronomy during almost 300 years. In the 14th and 15th centuries

almost every student must have known it. There are still more than 200 MSS left, besides a great number of variant editions and commentaries.

The Almagest among the Humanists and the Printers

Until well into the 15th century the Almagest appears to have been neglected by the great majority of scholars. Though they knew it by name, they cultivated astronomy by means of secondary texts. That Ptolemy's work was rescued from oblivion is the merit of a small group of scientists influenced by the new Humanism and its predilection for everything Greek, including the Greek sources of natural science. The central figure in this new development was a young Austrian scholar, George Peurbach (1423–1461) who spent most of his academic life at the university of Vienna[8]). Here the Humanistic movement had been inaugurated by the famous Italian scholar Aeneas Silvius Piccolomini (later Pope Pius II), who was Papal Legate in Austria 1443–1455 and, at least on one occasion, lectured at the university, where the masters gave lectures on classical subjects as early as 1451. Peurbach's humanistic proclivities appear from the fact that in the year 1454 he lectured both upon the Aeneid of Virgil, and on planetary theory. The latter lectures were made public in the form of a new manual called *Theoricae nouae planetarum.* This book was clearly intended to replace the old Theorica planetarum as a standard textbook. It is interesting to note that Copernicus presumably learned his first astronomy from Peurbach's book in 1491–96, when he was a student at Cracow, where Albert of Brudzewo had lectured upon it[9]). In the course of time it came to enjoy an immense popularity. Zinner (1938) mentions no less than 56 printings of the Latin text up to 1653, besides translations into French, Italian, and Hebrew. From the linguistic point of view the book would not seem to stem from a Humanistic pen, written as it is in the usual Latin idiom of Mediaeval astronomers. It also follows the old Theorica in the arrangement of the subject matter, and the general impression is that of a very careful exposition along traditional lines. Ptolemy is quoted in a few places towards the end of the book, but nothing indicates that Peurbach had made the Almagest the subject of any detailed study when he prepared his lectures. Despite his interest in classical literature he is not yet a Renaissance scholar in astronomy.

The following years saw a radical change in this situation, caused by the activities of two of the very greatest among the Byzantine scholars who had sought refuge in the West in the troubled years before Constantinople fell to the Turks in 1453. One of these was the Cretan philosopher George of Trebizond (1395–1484) who lived in Italy from about 1430 and became secretary to the Pope. For Nicholas V he produced a great number of translations from the Greek, among which was a complete Latin

8) There is still no satisfactory work on Peurbach. Most of what we know about him has been assembled by E. Zinner (1938).

9) Albert's lectures have been published by L. Birkenmajer: *Commentariolum super theoricas novas planetarum Georgii Purbachii ... per Mag. Albertum de Brudzewo,* ed. L. A. Birkenmajer, Cracow, 1900.

version of the Almagest published in 1451 (Incipit: *Peroptime mihi videtur, O Syre*). This was the first time Ptolemy's main work appeared in a translation from the original, if we disregard the less influential Sicilian version mentioned above.

But George of Trebizunt's translations were often too hastily made, and open to criticism by other scholars. Moreover, he was a militant Aristotelian who often offended against the prevalent Platonic sympathies of other Renaissance humanists. Finally, it is doubtful whether he possessed the necessary astronomical qualifications for producing a Latin Almagest. It would have fared better with a philologically competent astronomer as translator. That such a person could be found in Vienna was discovered in A.D. 1460 when the city was visited by another eminent humanist and papal legate, this time the famous cardinal Johannes Bessarion (1403–1472) who was at the very centre of the Humanistic movement in Rome; it was his private collection of Greek MSS which later became the kernel of the San Marco Library at Venice. Bessarion's antagonism to George of Trebizond was well known and resulted in 1469 in a treatise called *In calumniatorem Platonis*. But already during his stay at Vienna he tried to remedy the defects of the Almagest translation by persuading Peurbach to make a new translation from the Greek. Peurbach went to work with great eagerness and succeeded in finishing a draft translation or paraphrase of the first six books before his premature death in 1461. The unfinished MS was taken over by his pupil Johannes Müller, or Regiomontanus (1436–1476), who the same year followed Bessarion to Rome; later he travelled with the Cardinal to various places in Italy before in 1465 he became professor at the new university of Pressburg. There he stayed until 1471, when he finally settled down at Nuremberg.

Here at Nuremberg Regiomontanus created something which Latin Europe had never seen before – a scientific research institution outside the universities and independent of them. It was given generous support by a rich citizen called Bernard Walther (1436–1508) and comprised an observatory, a workshop for astronomical instruments, and last but not least a printing press for publishing scientific literature.

With this undertaking begins the history of the printed editions of the Almagest. An advertising sheet (Zinner, 1938, Pl. 26) published by Regiomontanus about A.D. 1474 announced that he had for sale the planetary theory of Peurbach, which thus became the first printed book on theoretical astronomy[10]). But the sheet also announced that Regiomontanus intended to publish a new translation of the Almagest: *Magna compositio Ptolemaei*; *quam uulgo uocant Almagestum noua traductione*. There is no doubt that Regiomontanus had the translation by Peurbach in mind from which we may conclude that he himself had the intention of finishing it. However, again an untimely death intervened, and the project came to nothing. Not until 1496 did a German printer at Venice publish the part of the work completed by Regiomontanus[11]).

10) *Theoricae nouae planetarum Georgii Purbachii astronomi celebratissimi*: *cum figurationibus oportunis*; 20 leaves in-fol., Nuremberg, ca. 1473: Ex officina Joannis de Regiomonte habitantis in Nuremberga oppido Germaniae celebratissimo.

11) *Epytoma Joannis de Monte regio in Almagestum ptolemei*, Venetiis (Hamman) 1496. – Other editions are Basle, 1543, and Nuremberg, 1550.

Thus the first attempt at making the fundamental work of astronomy accessible in a Latin printed version had produced nothing but a torso – an incomplete, free, paraphrase. At a time when the cry for printed editions of almost any classical author was heard all over Europe this was a serious situation, particularly since Ptolemy's other main work, the Geography, had already been on the market for many years, printed as early as A.D. 1475 by Hermann Lichtenstein at Vicenza, and later in 1482 (Ulm), 1486 (Ulm), and 1490 (Rome). Also the astrological Tetrabiblos had appeared in Latin as the *Liber quadripartitus*, printed in 1484 and 1493 (Venice), thus confirming the general rule that in Ptolemy's case the astrologer was always made known before the astronomer.

The honour of having published the first complete translation of the Almagest fell to the printer Petrus Lichtenstein, who in 1515 printed the old translation by Gerard of Cremona[12]). Thus the printed Almagest entered the world of the Renaissance in a Latin translation made from an Arabic version of the Greek original. In other words, it was the Almagest of the Mediaeval astronomers, and therefore apt to awaken the suspicion of any scholar of Humanistic inspiration. A translation directly from the Greek was much to be preferred, and in 1528 the famous Giunti printing house at Venice thought it proper to publish the unreliable Latin version made in 1451 by George of Trebizond[13]).

These two diverging versions made it imperative to have a printed edition of the original Greek text. It was prepared by Simon Gryneus and Joachim Camerarius from a (now lost) MS at Nuremberg formerly in the possession of Regiomontanus, and said to have been valued higher than a province by Bessarion. The edition appeared at Basle in 1538[14]). Thus, at long last, the original Almagest had been resurrected. But the two unsatifactory Latin versions remained the only translations for centuries to come.

The new editions made it possible to study the more technical aspects of the Almagest better than before at a time when the general level of mathematical knowledge was enhanced, partly due to a first rate textbook[15]) of plane and spherical trigonometry by Regiomontanus, published in 1533 long after his death. In a way the great work of Copernicus may be regarded as one of the principal results of this renewed interest in theoretical astronomy as a mathematical discipline, and for still a century to come Ptolemy kept his position as the greatest of all astronomers. On the isle of Hven Tycho Brahe had Ptolemy's portrait painted, and gave it an honourable place in his house above a Latin epigram of his own composition. As late as 1651 the Bolognese astronomer Giov. Batt. Riccioli (1598–1671) gave the title of *Almagestum*

12) *Almagesti Cl. Ptolemei Pheludiensis Alexandrini, astronomorum principis, opus ingens ac nobile, omnes cælorum motus continens*, Venetiis, 1515, in officina Petri Liechtenstein.
13) *Almagestum seu magnae constructionis mathematicae opus plane divinum Latina donatum lingua ab Georgio Trapezuntio*. In urbe Veneta, (Junta), 1528.
14) *Claudii Ptolemaei Magnae Constructionis, id est Perfectae coelestium motuum pertractationis, Libri XIII*, Basileae, (Joh. Walder), 1538.
15) *Doctissimi Viri* [...] *Joannis de Regio Monte de Triangulis omnimodis libri quinque*, Norimbergæ, (Petreius), 1533. – Facsimile edition with English translation: *Regiomontanus on Triangles*, by B. Hughes, Madison, Wisc. 1967.

novum to his immense handbook of astronomy – one of the most comprehensive ever written – and hailed Ptolemy as the Prince of Astronomers, Geographers, and Astrologers.

The Declining Fame of Ptolemy

But about this time, after a reign of almost 1500 years, Ptolemy's star began to fade. Copernicus had shown the mathematical possibility of a heliocentric theory of the Solar system, and even though the new theory used the same mathematical technique as the Almagest, many found it preferable. This was not only for philosophical reasons, but also because it explained the order of the planets which in the old astronomy had been arbitrary. Kepler had provoked a much more fundamental theoretical revolution by deducing his laws for the motion of the planets from Tycho's observations, and Galileo had finally, by means of the telescope, proved the impossibility of upholding the traditional physical notions of the nature of the heavenly bodies. These notions were mainly Aristotelian, but had become deeply connected with Ptolemy's name. But even a militant Copernican like Galileo held Ptolemy in high esteem, maintaining that both he and Aristotle would have been Copernicans if they had known the observations and other reasons which had moved Copernicus to change the system of the world (*Opere*, vii, p. 562, 1933).

Others were less judicious, and at last it became clear that the new astronomy not only meant the ruin of Ptolemaic planetary theory, but also pulled down his own scientific reputation. In the 18th century only a few had sufficient historical sense to realize that even if Ptolemy's astronomy had to be discarded, he himself had been a better scientist than most of those who now talked haughtily of his mistakes (that he was no Copernican) or his dogmatism (that he believed in natural circular motions). This degradation set in with the great French astronomers, among whom was Jerome le Français Lalande (1732–1807) who tried to rob him of any original significance: *One is convinced that Ptolemy was no observer, and that he took everything which is good in his work from Hipparchus and his other predecessors*, says Lalande in the sketch of the history of astronomy which he included in his *Astronomie* (Lalande, 1792, p. 119). From now on the myth of Ptolemy as a mere imitator of Hipparchus is commonplace in almost any account of the history of astronomy with the notable exception of Laplace (1796 p. 235). To Lalande, Ptolemy is important only because he has preserved some valuable observations of the ancients, and because the Almagest kept astronomy alive during the dark ages until the time when Copernicus revitalized it. But though it was thus impossible to rob Ptolemy of all astronomical importance, Lalande found it quite unforgivable that he should have tainted his memory with astrology: *One ascribes to Ptolemy a book on astrological predictions called Liber quadripartitus. However most critics find this to be unworthy of the learning and reputation of the author, the more so since in his Almagest there is nothing similar to that kind of phantasy* (Lalande, 1972 p. 119). This unsuccessful attempt to save Ptolemy from his own

personal belief in astrology reveals the curious lack of historical sense which was one of the characteristics of the Enlightenment. Lalande's evaluation of Ptolemy is clearly based, not upon any deep understanding of the role played by astronomy and astrology in Hellenistic society, but upon enlightened 18th century opinion of what an astronomer ought to do.

An even worse fate befell Ptolemy when Jean-Baptiste-Joseph Delambre (1749–1822) picked up the thread from Lalande. Delambre was not only a highly competent astronomer and one of the fathers of the metric system, but also one of the best informed among the historians of astronomy of all times. His *Histoire de l'Astronomie Ancienne* (1817) is still a mine of information on ancient astronomy. Nevertheless, Delambre seems to have been singularly blind when he had to evaluate Ptolemy himself. Thus he is angry with Ptolemy for his somewhat cavalier description of astronomical instruments: *In the first chapter of Book 5* ⟨of the Almagest⟩ *he assures us that he has observed the Moon with an astrolabe which he describes giving neither its radius nor the division in degrees. This is hardly the manner in which an astronomer describes an instrument which he has been using* (Delambre, 1817, p. xxvii). In the same way Delambre is dissatisfied with Ptolemy's description of his observations, particularly the measurement of the circumference of the Earth mentioned in Chapter 3 of the first Book of the Geography: *Is it thus one tells of new and important procedures – if they are real* (ibid., p. xxv)? The latter insinuation is no slip from the pen – Delambre makes it explicit in words allowing no ambiguity at all: *We can admit that Ptolemy has really seen these eclipses, but we cannot conclude that he was an observing astronomer. They give him a value of the eccentricity of the Moon so close to that which he deduces from the three Babylonian eclipses, that one is tempted to believe that his* ⟨own⟩ *three eclipses are computations made by means of tables* (ibid., p. xxvii). Or elsewhere: *Did Ptolemy observe? Are the observations which he claims to have made not calculations by means of tables, and examples intended for a better understanding of his theories* (ibid., p. xxv)?

This is without doubt the most serious accusation against Ptolemy ever made. Now he is no longer a rather insignificant follower of Hipparchus – he is no less than a scientific cheat, swindling with the very method of science and betraying the empirical character of astronomy, setting forth results computed from theory disguised as empirical data in support of this same theory. After that it is only small comfort that Delambre condescends to admit that Ptolemy had the sole merit of being a forerunner of Kepler, because he in his Lunar theory twisted one of the circles into an oval shape resembling a Kepler orbit: *One is permitted to believe that this hypothesis of Ptolemy has guided Kepler towards the ellipse* [. . .]. *Thus Ptolemy would have the honour of having prepared the way for Kepler, who again prepared it for Newton. This reflection, which so far as I know has not been made by any astronomer, will show that even if it seems that we would at times take some of his glory from Ptolemy and render it to Hipparchus, on the other hand we give him his due measure of justice, our only purpose being to write an exact history of astronomy* (Delambre, 1817, vol. ii, p. 381 f.).

Ptolemy Re-evaluated

One might think that Ptolemy's star would now have set never to rise again. Lalande and Delambre were authorities, and their attack was supported by an intimate study of Ptolemaic astronomy as a theoretical system. Yet their position suffered from one weakness which in the long run should undermine their conclusions. They wore the spectacles of their own century and were willing to condone the errors of Ptolemy only in so far as he could be regarded as a forerunner of one or another of the great pioneers of the astronomy of their own day. Therefore their judgment could not be final; it lasted only until historical research made it possible to see the Almagest and its author in their proper historical setting.

The main prerequisites for a better understanding of Ptolemy and his work were better editions than the old 16th century printings which Lalande and Delambre had to use. The first step in this direction was taken by a French scholar, the abbé Nicolas Halma (1756–1828), who at the beginning of the 19th century conceived a great plan of publishing a new Greek edition of the complete works of Ptolemy accompanied by a translation into a language with which every scholar in Europe was familiar, i.e. into French. This project was never completed, but it resulted in a new version and French translation of the Almagest[16]), and later of the important commentary to the same work by Theon of Alexandria[17]), besides the Geography and some minor works. Halma's preface to the Almagest is a curious but important document. His many excuses for undertaking this work are a testimony to the low ebb of Ptolemy's reputation at the time. Nevertheless, Halma succeeded in persuading Delambre to provide the new edition with a series of notes and explanations. Moreover, the preface contains the first serious study of the Almagest tradition based upon an intimate knowledge of all printed editions and a great number of manuscripts. In this respect it has preserved its value until to-day.

In a way Halma's text came both too late and too early; it was too late in so far as Lalande and Delambre had given their final judgment before having access to a good text. But it was too early because classical philological scholarship had not yet learned to deal with old texts and to make critical editions from a survey of all the manuscripts in the manner which emerged during the latter half of the 19th century. The need for such an edition was felt more and more as time went on, and towards the end of the century Johan Ludvig Heiberg (1854–1928) was commissioned to make a new, critical, and complete edition of Ptolemy to be published by Teubner in Leipzig. This project was not finished. In 1898–1903 Heiberg published the Almagest in Greek in a text which is now considered authoritative, and continued with a volume of *Opera minora* (1907), when his death intervened. The result is that we still have neither

16) *Composition Mathématique de Claude Ptolémée, trad. pour la première fois du grec en français sur les manuscrits originaux de la Bibliothèque Impériale de Paris,* par Halma, et suivie des notes de Delambre, I–II, Paris, 1813–1816.

17) *Commentaire du Théon d'Alexandrie sur le livre premier de la composition mathématique de Ptolémée,* traduit pour la première fois du grec en français par Halma, I–II, Paris, 1822–1825. Cf. the critical edition of Books I–IV by A. Rome.

a complete version of Ptolemy's works in the original Greek, nor a complete translation into a modern language.

Translations of the Almagest itself are also scarce. In French we are still left with Halma's old translation. In German there is a very useful and exact translation made from Heiberg's text by K. Manitius (1912–13) who also wrote a number of notes of great importance for the understanding of the astronomical contents of the text. A reprint of this translation with a number of corrections by O. Neugebauer was published in 1963 with a curious advertisement from the publishers, stressing the importance of Ptolemy in the times of space research . . .[18]). The first English translation (by Taliaferro) appeared in 1952.

The modern preoccupation with the Almagest in a reliable text and a good translation is the background of the re-evaluation which is now taking place. Ptolemy is no longer considered a mere compiler of earlier results in astronomy, but as a highly competent and original astronomer responsible for some of the most impressive features of planetary theory as set forth in the Almagest. One important branch of research is therefore concerned with the problem of separating his original contribution from that of Hipparchus and others. Another investigates the question of to what extent Ptolemy was influenced by Babylonian astronomers whose observations he certainly knew and made extensive use of, although inside an entirely different theoretical framework (Neugebauer, 1951, p. 161).

On the other hand the problem of the nature and authenticity of Ptolemy's observations has found no definite solution despite a considerable amount of work. But everything considered there is no doubt that today Ptolemy appears in a better light than in the days of Delambre. For B. L. van der Waerden (1954, p. 200) he marks the culmination of Greek astronomy. Similar voices have been raised by Sundman (1923), Okulicz (1933), and Rome (1938). There have even been attempts to show that he was one of the principal exponents in Antiquity of what can be somewhat summarily described as the scientific method (Kattsoff, 1947). And quite regardless of how Ptolemy's sincerity and integrity is evaluated there is no doubt whatever that he remains forever one of the most prominent figures in the history of astronomy, and that – in the words of O. Neugebauer (1951, p. 3) – *One cannot read a single chapter in Copernicus or Kepler without a thorough knowledge of the Almagest.*

18) »Die Zahl der Interessenten an dem Werk ist nicht zuletzt auch durch die Entwicklung der Weltraumforschung ständig gestiegen«.

CHAPTER 2

Physics and Philosophy in the Almagest

The Almagest begins with a preface [I, 1; Hei 1, 4][1]) in which the whole work is dedicated to a certain Syrus, about whom we know nothing. He must have been rather closely connected with Ptolemy, who dedicated other works to him also, thus the Planetary Hypotheses, the Handy Tables, and the Tetrabiblos, i.e. the main body of his astronomical and astrological writings.

This introductory chapter is of considerable importance from a philosophical point of view. In general, the Almagest is a highly technical and closely reasoned account of mathematical astronomy, written in a style which but rarely reveals anything about the personality of its author and his more philosophical opinions. The subject matter is everything and the author stands back all the time to let it appear in as objective a light as possible. In fact, the preface is one of the very few sections of the work in which we get more than a glimpse of Ptolemy's ideas on such general questions as epistemology, the classification of the sciences, the human value of scientific research, and the ethical implication of the study of astronomy.

The Classification of Knowledge

Let us first consider the Ptolemaic classification of the various kinds of knowledge. It can be illustrated by the following scheme:

$$\text{philosophy}\begin{cases}\text{practical}\\[2ex] \text{theoretical}\begin{cases}\text{theology}\\ \text{mathematics}\begin{cases}\text{arithmetic}\\ \text{geometry}\\ \text{astronomy}\end{cases}\\ \text{physics}\end{cases}\end{cases}$$

Here the first major division is between practical and theoretical knowledge. Ptolemy shows that this distinction is essential by means of an example: A man may have great insight in moral questions, and even extend it through his everyday experience of life, without any specialized education in ethics; but it is impossible to acquire any knowledge of the universe without theoretical studies in astronomy. How important this distinction was to Ptolemy appears from the fact that the Preface opens with a

1) This means Book I, Chapter 1, Heiberg's edition vol. 1, page 4. All references to the *Almagest* will be given in a similar way.

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_2,

praise of the 'genuine philosophers' who first noted it. These philosophers are not mentioned by name, but the distinction between practical knowledge resulting in human skill in arts and crafts, and theoretical knowledge leading to intellectual understanding is a commonplace among all philosophers of the Peripatetic School both in Antiquity and the Middle Ages. Therefore, the commentator Theon of Alexandria (ed. Rome, p. 321) has no difficulty in tracing it back to Aristotle himself, who adverts to it in many of his works (e.g. *Metaph.* vi, 1, 1025 b ff.). However, it is worth noticing that Ptolemy does not include the third Aristotelian category – poetic knowledge – in his scheme, although he was deeply impressed by the universal truth, the aesthetic function, and the emotional value of astronomy.

Since Ptolemy had a marked predilection for theoretical knowledge, we must now consider his division of this branch into theology, mathematics, and physics. Here again he followed Aristotle, who deduced these categories by his theory of abstraction. At the bottom of the scale we find 'physics', or natural science, which was defined by Ptolemy as the study of the ever changing material world. For Ptolemy physics was concerned with questions like e.g. whether a material substance is hot, or cold, or sweet, etc. Most of its objects belong to the corruptible part of the universe inside the sphere of the Moon. This is, roughly speaking, in agreement with Aristotle, who defined physics as the study of 'nature', and nature as the principle of 'motion' or change (*Phys.* ii, 1, 192 b, cf. *Metaph.* vi, 1, 1025 b). This implies both that physical objects are material, and that most of them (all apart from the heavenly bodies) are subject to generation and corruption. What Ptolemy omits to mention – but certainly maintains – is that in physics such objects are studied with particular regard to their materiality.

At a more abstract degree of knowledge we find the science of mathematics, which to Ptolemy is an investigation into the nature of the forms and motions of material bodies, implying notions like figure or shape, quantity, magnitude, space, and time. Again this is in agreement with the Aristotelian doctrine (*Phys.* ii, 2, 193 b) of mathematics as an abstract science of the physical world, studying the same bodies as one does in physics, but without regard to their constituent matter and concentrating on their 'mathematical' properties.

It is one of the main tenets of Aristotelian epistemology that it is possible to carry the process of abstraction even further and to study the world under still more general points of view in terms of being, existence, cause, effect, and similar 'metaphysical' concepts. The science resulting from this most general kind of inquiry is called theology both in Aristotle (*Metaph.* vi, 1, 1026 a) and in Ptolemy. In the former it is crowned with a proof of the existence of a Supreme Being called God and conceived as the 'Prime Mover' of the whole universe. In another place it is defined more directly as the knowledge of the invisible, immaterial, and unchanging God (*Metaph.* iv, 8, 1012 b, cf. xii, 7, 1072 b). Ptolemy also infers the existence of God in this philosophical sense from a metaphysical argument: The senses are incapable of analyzing the phenomena of the material world into their constituent matter, forms, and motions. This can be accomplished only by reason, with the result that reason not

only shows us motion as something different from matter and form, but also reveals an ultimate cause of all motion, i.e. God.

On Certainty in the Various Fields of Knowledge

The three main branches of theoretical knowledge have not the same epistemological status. In his Preface Ptolemy is concerned primarily with the degree of certainty which they provide. Here he finds characteristic differences which to him are sufficient motivation for his personal preference for mathematics as the most perfect discipline. Like Plato, Ptolemy is deeply impressed by the perfect character of mathematical truth. Theology has no similar certainty, because its object is absolutely invisible and incomprehensible. Physics, too, is unable to attain absolute truth because of the corruptibility and obscurity of matter. This is why there is no hope that philosophers will ever agree, or arrive at a common opinion on these two branches of knowledge.

It is obvious that here Ptolemy is more agnostic than Aristotle. To the latter theology was the same as metaphysics, and he would surely have maintained that metaphysics is able to analyze the whole realm of existence in terms of metaphysical relations of an absolutely true character. But to Ptolemy theology is defined by means of the concept of God as a transcendent being beyond human comprehension, – with the corollary that theological statements are less certain than those of mathematics. Moreover, the reference to the disagreement between philosophers reveals that Ptolemy belongs to a later age than Aristotle. In fact, they were separated by a period of five hundred years in which one conflicting school had succeeded another and many different opinions on the nature of God had been ventured. This explains Ptolemy's reticence towards theology. He is certainly no unbeliever, as both the preface and several other places in the Almagest show. But he cannot ascribe to theological statements the same epistemological quality as to mathematical theorems. As we are going to see, this view entails the consequence that Ptolemy warns us against too much anthropomorphism in science. Before entering upon the very complicated theory of latitudes in Book XIII (see page 355) he tries to refute the opinion of those philosophers who maintain that astronomers ought to construct simple theories only, since the simplicity of the Supreme Being should be reflected in the description of His works. Ptolemy reminds us, first, that philosophers are not agreed upon what 'simple' means, and second, that what is simple to God may appear far from simple to man. Simplicity in God does not entail 'simplicity' in the description of nature.

In the same way physics has to be put in a more humble position for the reason that its subject matter – the material world – is both obscure and corruptible. The obscurity of matter is a commonplace in all Greek philosophy and it is no wonder that Ptolemy subscribes to this idea. It is a little more curious that he refers also to the corruptibility of matter as an obstacle making absolute truths about the physical world impossible.

Ptolemy's Conception of Mathematics

It would seem that the reason for this opinion is a conception of science slightly different from that of Aristotle to whom a true scientific statement has the character of being invariably true at all times and under all circumstances. Thus the statement that heavy bodies have a natural tendency to move towards the centre of the world is always true regardless of all the contingent vicissitudes of actual heavy bodies. So the existence of change and corruptibility in the physical world does not exclude the possibility that there may be invariable relations among its ever changing things, expressed in eternally true statements. Here Ptolemy seems to disagree when stating in the preface that *astronomy alone is concerned with the investigation of a world remaining the same throughout all eternity* [...] *which is a characteristic property of science* [I, 1; Hei 1, 6]. In other words, to Ptolemy, eternally true statements can only be about eternally unchanging objects.

We must now examine Ptolemy's conception of mathematics a little closer. First, mathematics is a very general science. It can be grasped through the senses or without them, just as there is a mathematical aspect of any kind of material being – corruptible as well as incorruptible. Perishable substances change with their changing forms and are thus objects of mathematical study, just as well as the unchanging forms of eternal beings of an ethereal nature, i.e. the heavenly bodies.

Next, mathematics leads to absolutely certain truths which, once established, can never be subjected to doubt. This is because mathematical truth is acquired by means of logical proofs, regardless of whether they are concerned with arithmetic or geometry. It is this logical character which conveys a certainty never to be attained through the testimony of the senses, and which is foreign to physics. Ptolemy confesses that we here have his own deepest personal motive for becoming a mathematician.

Mathematics, Astronomy, and Astrology

One could think that this might have disposed him to become a 'pure' mathematician. He did, in fact, a small amount of theoretical research in this field (see page 47) but the preface does not conceal that astronomy was his main interest, at any rate when he wrote the Almagest: *This is the reason which has moved us to devote ourselves – to the best of our abilities – to this pre-eminent science in general* [i.e. mathematics], *but particularly to that branch of it which is concerned with the knowledge of the Divine and Celestial Bodies,* after which follow the words quoted above on the unchanging nature of the heavens and the ensuing timelessness of astronomical truths.

At this point there arises a difficulty: How could Ptolemy become an astronomer if only mathematics was able to satisfy his longing for eternal truth? This question seems to be connected with an apparent ambiguity in Aristotle's opinion on where astronomy should be placed in the hierarchy of knowledge. In the *Physics* (ii, 2, 194 a) he calls astronomy *rather physical than mathematical* and compares it with optics and

harmonics (i.e. the theory of music). But in the *Metaphysics* (xii, 8, 1073 b) astronomy appears as *one of the mathematical sciences nearest to philosophy*. In Ptolemy so much is clear that astronomy is not tainted with the obscurity and uncertainty ascribed to physics, notwithstanding the fact that it is concerned with the heavenly bodies. These bodies have a material character both to Aristotle and Ptolemy, although their matter is of a particular, ethereal nature. Nevertheless, this does not prevent astronomy from being of a loftier status than the other natural sciences.

The solution of this puzzle seems to be that Ptolemy drew a different boundary between mathematics and physics from Aristotle's. It is not said in so many words, but it is as if he thinks as follows: If one studies the changing and corruptible material world the resulting science belongs to physics, even if it has a mathematical form. On the other hand, a study of the unchanging and eternal heavens of a similar mathematical form belongs to the science of mathematics.

If this is the correct interpretation, it is clear that Ptolemy's classification of science rests upon a different foundation from that of Aristotle. The latter had defined the limits of physics, mathematics, and metaphysics (or theology) by means of the formal objects only of these sciences; and since physics studies natural objects in their material form he was compelled to include any natural science in physics, even if its statements were expressed by means of mathematics. Ptolemy goes a step beyond Aristotle in using the material object of science also as a basis for classification. It is the different properties of celestial and terrestrial matter which enable him to lift astronomy out of the realm of physics into the domain of mathematics.

It should be noticed that at this particular point he had no great following among ancient philosophers of science, who in general kept to the more logical and unambiguous Aristotelian principle of regarding the formal object only. Later, in the Middle Ages, the difficulty was felt anew and gave rise to the doctrine of *scientiae mediae*. These are sciences which, like astronomy, optics, harmonics, and mechanics, are concerned with a mathematical description of the material world. Accordingly they have something in common with both physics and mathematics without being totally subsumed under either of these headings[2]).

The status of astronomy as a part of mathematics is also, perhaps, the explanation of the very remarkable fact that the Almagest is completely free of astrology. This is not because Ptolemy did not believe in the possibility of making predictions from the stars. On the contrary, his Tetrabiblos is one of the most comprehensive manuals of judicial astrology ever written (see page 400). But astrology is concerned with the influence of the celestial world upon the terrestrial, and the influence of the stars is very closely connected with their physical nature. This means that in astrology one cannot abstract from the material qualities of the stars; for that reason astrology must be classified as a part of physics, and not treated on a par with mathematical

2) Thus St. Thomas Aquinas maintains that *some of the sciences applying mathematics to natural phenomena are placed in between, for instance music and astronomy. However, they are more related to mathematics because what interests the physicist in them is rather the material aspect, but what interests the mathematician is more the formal aspect. – In Libr. Boethii de Trinitate* q. 5, a 3, ad 6^{m}.

astronomy even if it makes use of mathematical computations. This is true without regard to the fact that in popular terminology a 'mathematician' often means an astrologer, as we see e.g. from St. Augustine's fulminations against them (*De civ. Dei* v, 5, but cf. *De Trin.* ix, 6).

The Human Value of Astronomy

The special status of astronomy makes it useful to other sciences. Thus it helps theology because it draws our attention towards the celestial world and helps us to discover the First Cause of its motion – the Prime Mover being God Himself: *It opens the road to the Divine world by imparting knowledge of a force which is eternal and different from every other. It alone is able to discover the relations between eternal substances not subject to any influence, and the sensible, moving and movable world, through the phenomena, order, and disposition of their motions.*

This argument follows the Aristotelian proof of the existence of God as the Prime Mover. When theology in the Middle Ages rose to the top of the hierarchy of knowledge, the argument explained also why astronomy was given the most honourable position among the various sciences. Ptolemy himself comes very near to the same attitude when he maintains that the study of astronomy has ethical consequences and is a step on the road to human perfection: *More than any other thing it contributes to make us better, making us more aware of what is good and beautiful in the moral life. For those who study this subject find a harmony between the Divine things and the beautiful order of the propositions. This makes them love this Divine Beauty and makes them accustomed to take it as a model of their conduct.* Thus the harmony and beauty of the celestial world become a kind of second nature in those who study it. If this is a testimony of personal experience, and not only a repetition of an old Platonic idea from the *Timaios* (90 c), it certainly reveals something of the spiritual life of Ptolemy himself.

Also physics may profit from mathematics and astronomy, since the particular properties of material bodies are revealed through their characteristic motions which are described mathematically. Thus the natural motion of

the corruptible is	along a straight line
the incorruptible is	circular
the heavy and passive is	towards a centre
the liquid and active is	away from a centre

This is traditional Aristotelian physics. But Ptolemy proves to have been influenced by later schools of thought when he characterizes heaviness as a passive, and lightness as an active quality of matter. This was one of the main features of the physics of the Stoics[3]).

3) On the influence of Stoic philosophy upon Ptolemy see Lammert (1922). On active and passive qualities, see Galen, *Introd. s. medic.* 9, quoted by Sambursky (1959) p. 119.

This is, therefore, the final purpose of the Almagest: *We shall always strive to augment this Love of the Science of eternal things, not only by making us acquainted with the achievements of those who have studied these things before us, but also by contributing a little ourselves with discoveries made in the brief interval since then. Thus we shall seek to describe briefly every important contribution from our own time* [....]. *Finally, in order to attain the purpose of this work we shall in a suitable order comment upon anything useful to the theory of the Heavens. In order to make the exposition as brief as possible we shall only summarize that which has already been ascertained by the Ancients, concentrating upon problems not sufficiently dealt with or proved before.*

The Plan and Disposition of the Almagest

After the Preface follows a short chapter [I, 2; Hei 1, 8 ff.] in which the general plan of the Almagest is outlined. It will contain

A. A brief general part dealing with the Earth and its position in the universe as a whole [I, 3–11].

B. A long special part comprising 3 sections:

1) Spherical-astronomical problems and their mathematical solutions [I, 12–II];
2) the theory of the motion of Sun and Moon [III–VI] as a necessary preliminary to the theory of the planets;
3) the theory of the stars, divided into two sub-sections
   a) the fixed stars [VII–VIII],
   b) the five planets [IX–XIII].

Section B 3 is said to be the core of the whole work (perhaps most of it is due to Ptolemy himself whereas much of sections 1 and 2 was known to his predecessors, in particular Hipparchus). The method of exposition must everywhere be to start with the most certain observations to be found in previous or contemporary observers; then the next step must be to connect such series of observations by means of a mathematical theory, developed in the form of geometrical models.

This programme is in perfect agreement with what Greek astronomers had been used to doing for centuries, and contrasts clearly with the purely algebraic theoretical structures applied by their Babylonian colleagues. But – as we are going to see – Ptolemy does not conceive his method as a purely inductive process in which theories are created from observations in a more or less automatic way. On the contrary, Ptolemy usually adopts a manner of exposition in which the general features of the final model are presupposed or postulated already at the beginning. Occasionally this style makes the Almagest difficult: a model is going to be built up upon empirical data described in terms of the model itself. Therefore, the ordinary structure of a chapter in the Almagest is composed of 4 parts:

1) a brief, qualitative sketch of the series of phenomena to be explained;
2) a preliminary account of the geometrical model as a postulate without proof;
3) a meticulous deduction of the parameters of the model from carefully selected and recorded observations, often by means of reiterative methods;
4) a verification that with these parameters the model really explains the phenomena in a quantitative way.

On the other hand this method may be regarded as a historical support of the view that a purely inductive procedure in science is impossible – a certain amount of theory has to be present already at the beginning as an indispensable framework for the selection and description of the 'relevant' empirical data.

The Nature of Astronomical Theories

Before tackling the problem which interests him most of all, viz. how to give a theoretical account of the motions of the Sun, Moon, and planets, Ptolemy has to decide upon a number of methodological and other philosophical questions of a more general nature than those touched upon in Book I (cf. page 26).

These questions are not dealt with in a really systematic way. Accordingly we have to extract them, first from a number of occasional remarks scattered all over the Almagest, and second, from an investigation of the methods actually employed by Ptolemy for constructing planetary theories. In other words, we must compare what Ptolemy says he is going to do with what he actually does. This will give us a much deeper insight into his philosophy of science than the very general considerations found in Book I.

The most obvious feature of this philosophy is Ptolemy's deep conviction that things are not what they seem. The real structure of the Universe is hidden from our bodily senses and can be revealed only through the eyes of our mind. As deeply as any Platonist, Ptolemy must accordingly distinguish the observable phenomena from their hidden causes. This is only a particular consequence of the common Greek idea that what really exists must be without change, so that all changing phenomena must be mere appearances. On the other hand, science is founded upon the belief that even the most disorderly events in nature are caused by immutable laws to be uncovered by the scientist.

This general attitude has immediate implications for astronomy. Already in the preface to Book I Ptolemy calls attention to the perfect order and unchanging beauty of the heavens which had drawn his inclination towards the study of the stars in their courses. An obvious example of this order is the perfect regularity with which the heavens perform their diurnal rotation. But this is a somewhat particular case, and in general such beautiful regularities do not belong to our immediate experience.

This is most easily seen in the five planets whose normal motions towards the East among the fixed stars are sometimes interrupted by retrograde movements towards the West. Even the most perfect heavenly body, the Sun itself, is not untainted by

disorder; for even if it has no retrograde motion, the unequal lengths of the four seasons show that it performs its course along the ecliptic with a changing velocity. Thus most celestial phenomena exhibit an intriguing mixture of order and disorder, forcing us to admit that what we perceive by our senses are not the real motions, but only appearances.

Therefore, the task of the theoretical astronomer is clear: by means of a suitable intellectual effort he must try to prove beyond doubt that even the most confused and disorderly celestial phenomena can be explained in terms of invariable, orderly laws[4]. Accordingly his aim must be first to discover and to formulate such laws, and second, to demonstrate that the phenomena can be deduced from them. We know already that in order to cope with this problem we must proceed by way of mathematics. In Greek astronomy – as opposed to Babylonian calculations – this is the same as to say that astronomical theories must be expressed in geometrical terms.

From the very infancy of Greek science one particular geometrical concept had been proposed as the principal conceptual tool of the theoretical astronomer. If we are to believe Geminus (*Elem. Astr.* i, 19) it was the Pythagoreans who first *assumed that the motions of the Sun, Moon, and planets are circular and uniform.* What this means is clearly stated in the Almagest [III, 3; Hei 1, 216] where Ptolemy considers a point moving upon the circumference of a circle in such a way that a line from the centre to the moving point describes equal angles in equal times. This is equivalent to saying that the point moves with a constant angular velocity, a term of more recent origin and unknown to Ancient or Mediaeval astronomers.

This notion of uniform, circular motion is fundamental. In fact, the history of Greek theoretical astronomy is, to a very large extent, the history of a long series of efforts at explaining away the observed irregularities in the heavens by resolving even the most complex celestial motions into a set of uniform, circular components. Here Aristotle had advocated a very strict position, admitting only component circular motions which were

1) uniform as seen from their own centres, and
2) concentric with the universe as a whole.

As is well known, a theory fulfilling these requirements was proposed by Eudoxos (*Metaph.* xii, 8, 1073 b), but it failed because it was unable to account for the varying distances of the planets (cf. Herz, 1887, and Neugebauer, 1953). Consequently both Apollonius and Hipparchus were forced to abandon the second condition, admitting into their theories circles with centres other than that of the universe. On the other hand they retained the first condition, using only motions with constant angular velocities relative to their own centres.

Ptolemy's position is rather complex. We notice first that every time he begins his investigation of the theory of a particular planet, he explicitly acknowledges the validity of the principle of uniform, circular motion. Thus in the introduction to the

4) This is the famous program of *saving the appearances,* cf. Mittelstrass (1962) and Wasserstein (1962).

Solar theory [III, 3; Hei 1, 216] he says that in order to *explain the apparent irregularity of the Sun it is necessary to assume in general that the motion of the planets is uniform and circular*. The reason given for this dogmatic statement is metaphysical, the principle being said to be the only one *consistent with the nature of Divine Beings* (i.e. the planets) [IX, 2; Hei 2, 208]. Consequently, the construction of a theory along these lines is *a great thing, the accomplishment of which is the goal of a philosophically founded mathematical science* [ibid.]. However, it is also *an enterprise which for many reasons is connected with great difficulties.*

In IX, 2 Ptolemy enumerates some of these difficulties. However, already at this place it should be noted that one of the most intriguing features of his planetary theories is passed over in silence. Ptolemy always claims that he adheres to the principle of uniform circular motion without any formal modifications. But, as we shall see, this is mere lip service to a venerated traditional dogma from which Ptolemy does not hesitate to depart when he finds it opportune to do so. In fact, he takes an important step beyond Hipparchus and all earlier astronomers by admitting into planetary theory also circular motions which are uniform only as seen from a point other than the centre of the circle (see later page 277)[5]. Consequently they are non-uniform with respect to the centre. For this departure from a well-established tradition Ptolemy was often blamed by later astronomers[6]. This is, of course, unjustified from a purely astronomical point of view, since a theoretical science cannot be confined by assertions of a non-scientific character. But it is, nevertheless, a little disappointing that Ptolemy everywhere pretends to stick to tradition without commenting on the points where he discards it.

The General Part of the Almagest

The Preface to the Almagest considered on the preceding pages contained, as it were, Ptolemy's general philosophy of science. We shall now examine what he calls the 'General Part' of the Almagest, i.e. the 6 chapters [I, 3–8; Hei 1, 10–31] in which he describes the heavenly sphere, the various motions observed in the heavens, and the shape, position, size, and immobility of the Earth. In general, the doctrines here described characterize what has become known as the Ptolemaic Universe, although they stem, all of them, from a much earlier period in Greek Natural Philosophy, and were first described in a coherent manner by Aristotle. This is the reason why Ptolemy is able to deal very briefly with them. But it should be remembered that in the *Planetary Hypotheses* he gave a personal contribution to this cosmology, attempting, among other things, to compute the distances of the planets from the Earth, and the size of the whole universe. These latter problems will be investigated in a later chapter

5) This means that it is a mistake to consider Ptolemaic methods as a kind of rudimentary Fourier analysis, cf. Aaboe (1960 b).
6) See for instance Ibn al-Haitham's critique, in Pines (1962), and also Copernicus, *De rev*. Praef. iii b.

(page 391 ff.). At present we shall consider only those cosmological opinions which Ptolemy found it necessary to include in the Almagest as prerequisites to the understanding of planetary theory.

The Diurnal Rotation of the Heavens

The third chapter of Book I [I, 3; Hei 1, 10] begins with some speculations on how the Ancients might have come to the view of the heavens as a rotating sphere. All the time they saw the Sun, Moon, and stars rising above the Eastern horizon, culminating in the middle of the heavens, and setting beneath the Western horizon, moving continually upon parallel circles with a common centre at the Northern pole of the heavens. The fact that the nearer a star is to the pole, the smaller its circle will be, leads to the idea of a rotating sphere upon which they are placed.

In fact, any other idea will be unable to save the phenomena. Thus Ptolemy turns down the conception that all stars perform rectilinear motions through an infinite space, a view perhaps advocated by Xenophanes of Colophon (about 570–475 B.C. (Kirk and Raven, p. 173)) but ascribed by Theon (ed. Rome, p. 338 f.) to Epicurus (about 342–271 B.C.), which is wrong, since Epicurus speaks clearly of the rotation of the heavens (Diog. Laert, X, 92). This hypothesis would explain the phenomena of rising and setting, but not the fact that e.g. the Sun and the Moon retain their size and even look a little larger at the horizon than higher up. If they moved along an infinite straight line they would rise and set as vanishing points. Furthermore, it seems impossible to explain the impression that it is the same stars which return at the same places night after night.

Another critique is directed against those who like Xenophanes maintain that the stars arise from the Earth and are kindled only to extinguish when they fall to the Earth again. This idea is ridiculed by Ptolemy from the fact that the Earth is spherical; this implies that when a star is rising for one observer it will be setting for another, which makes this theory impossible.

Ptolemy continues with the assertion that any other figure of the heavens than that of a sphere would change the relative distances of the stars during their daily revolution, thus making the constellations change contrary to what experience shows.

A further argument is that if the heavens were not a sphere sundials would not show the correct time – presumably a reference to the hemispherical sundial, or *polos* (Herodotus, ii, 109; cf. Dicks, 1954 p. 77) used in various forms by the Babylonians, Greeks, and Romans.

To these astronomical reasons for the spherical shape of the heavens are added two others, one mathematical and the other physical. First, the sphere is that among all geometrical forms which has both the smoothest and easiest motion, and the greatest volume relative to its size. The latter is the isoperimetric property of the sphere, already known to Zenodorus, who is difficult to date, but certainly lived before Ptolemy's time (Theon, ed. Rome, p. 360 ff.; cf. v. d. Waerden, 1954, p. 306).

The physical argument rests upon the assumption that the celestial matter, or ether, is composed of small particles or molecules. These are finer and more regular than those of which other substances are composed, and must therefore be spherical. A further assumption is not stated explicitly in the Almagest, but is formulated by Theon: *any physical body consisting of similar parts will have the same figure as these parts.* Therefore, the whole ethereal body, i.e. the heavens, must be spherical (Theon, ed. Rome, p. 49). A further conclusion drawn from this principle is that the individual heavenly bodies (the stars) must also be spherical. The latter is confirmed by the more acceptable assertion that otherwise they would not appear in the same shape to observers at different places of the Earth. This would be a good argument if Ptolemy did not, at other places, deny any observable parallax of the heavenly bodies apart from the Moon (and, perhaps, the Sun).

The Spherical Shape of the Earth

That not only the heavens, but also the Earth is spherical is proved by a series of arguments based on astronomical facts [I, 4; Hei 14]. Here Ptolemy begins by considering the times at which the same Lunar eclipse is observed by different observers. The important thing is that a Lunar eclipse is an objective phenomenon in the sense that the entrance of the Moon into the shadow of the Earth happens at a definite instant, independently of where the observer is situated. Now it is a fact that the local time of, say, the beginning of an eclipse is not the same to all observers. The event happens later the further the observer is situated towards the East, the time difference being proportional to the distance. This shows that the Earth has a curvature relative to the East–West-direction.

Furthermore, this curvature makes the Earth a convex body. Were it concave, i.e. hollow, the stars would rise earlier for a Western observer than for an Eastern. Were it flat, they would rise simultaneously for all observers (Figure 2.1).

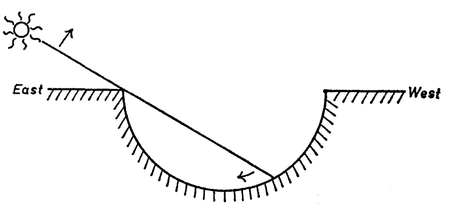


Fig. 2.1

Now one could imagine that the Earth was a circular cylinder with its axis in the North–South direction; this would be consistent with the curvature proved above, but not with the fact that the height of a star above the horizon will change as the observer travels North; this implies that his horizon (the tangential plane to the Earth) has no fixed direction relative to the axis as it would have on a cylinder (Figure 2.2).

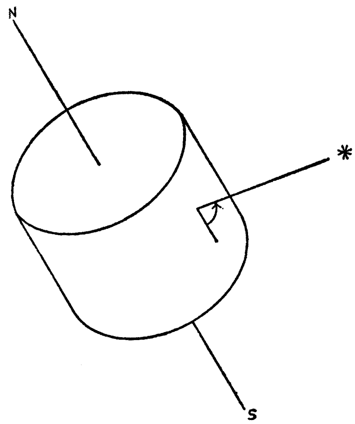


Fig. 2.2

Furthermore, travelling North we observe how stars gradually disappear forever from the Southern sky, at the same time as more and more stars become circumpolar in the Northern. On a cylinder the same stars would be either visible or invisible without regard to the position of the observer (Figure 2.3).

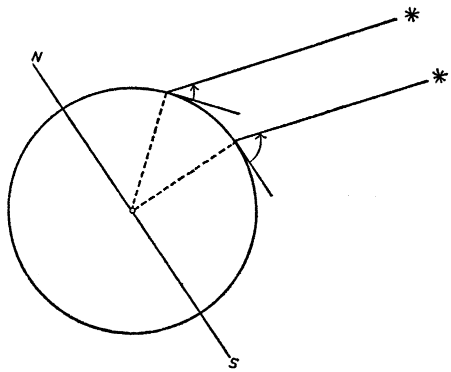


Fig. 2.3

The result is that the Earth must have a double curvature. Finally, Ptolemy draws attention to the well known experience that an observer on board a ship approaching the coast will see first the top of the mountains, then gradually more and more.

It is interesting to compare this chapter with the classical text on the spherical shape of the Earth in Aristotle's work *On the Heavens* (ii, 14, 297 a). Here Aristotle begins with a series of physical arguments concerning gravitation, having previously proved the Earth to be at the centre of the universe. This latter opinion is first established by Ptolemy in the following chapter. The proofs based on theories of gravitation have

to wait, and he is left with purely astronomical arguments. This gives his treatment more unity and coherence than the corresponding chapter in Aristotle. On the other hand one cannot help wondering why Ptolemy ignores one of the most striking proofs found in Aristotle, viz. that during a Lunar eclipse the Earth casts a circular shadow on the surface of the Moon (*De caelo* ii, 14, 297 b).

The Central Position of the Earth

So far Ptolemy has established that the Earth is spherical and placed somewhere inside a spherical universe. Now follows a proof that its only possible position is at the centre of the heavenly sphere [I, 5; Hei 1, 17]. The proof is indirect since Ptolemy shows that any other position is impossible for astronomical reasons, leaving out any 'physical' argument derived from the theory of gravitation.

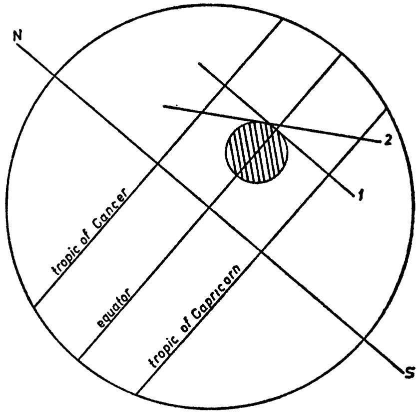


Fig. 2.4

1) If the Earth were placed outside the axis but at the same distance from each pole (i.e. in the plane of the celestial equator, see Figure 2.4) there would be trouble with the equinoxes. On the terrestrial equator (position 1 of the horizontal plane) day and night could never be equal everywhere simultaneously, since the horizontal plane would divide both the celestial equator and any other diurnal arc of the Sun into unequal parts. At other places (position 2) either the same would be true, or else the equinoxes would not fall in the middle between Summer and Winter solstice. This is because the circle halved by the horizon would no longer be the celestial equator, but some other circle lying asymmetrically with respect to the two tropics.

Moreover, a traveller moving East would see the relative distance of the stars change, and the constellations with them. He would also find the time between the rising and culmination of a star at one place different from the time between culmination and setting at another place on the same parallel.

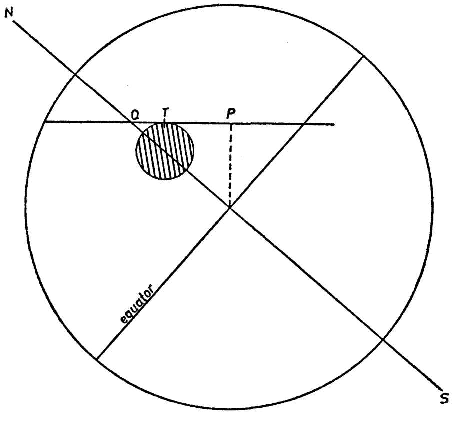


Fig. 2.5

2) Next we suppose the Earth to be placed on the axis but nearer to one pole than to the other (Figure 2.5). In this case an observer at a point T outside the terrestrial equator will find the heavenly sphere divided into two unequal parts by his horizontal plane. In particular the ecliptic would be divided into unequal parts too, and we would not always see 6 zodiacal signs above and 6 below the horizon, which is contrary to experience.

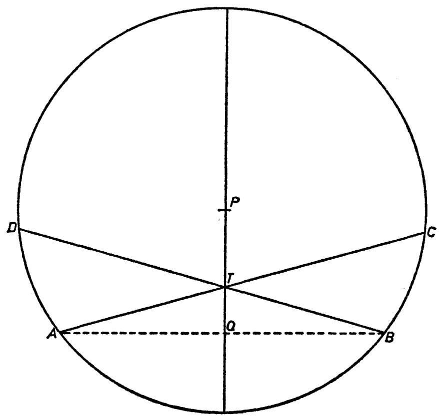


Fig. 2.6

Another consequence would be that at the equinoxes (if such dates exist for the given position of the Earth), the Sun would neither rise in the East nor set in the West. This argument is not made explicit. Figure 2.6 shows the horizontal plane of Figure 2.5 The dotted line AQB is the trace of the plane of a circle parallel to the equator and halved by the horizontal plane. If this circle is so situated that the Sun is able to reach it, then there will be a date when day and night are equal. Looking at the horizon

from above, the trace of this circle will be the line AQB. It follows that a gnomon at T will cast a morning shadow along TC and an evening shadow along TD; these two shadows are not diametrically opposed, contrary to what is observed everywhere on the Earth.

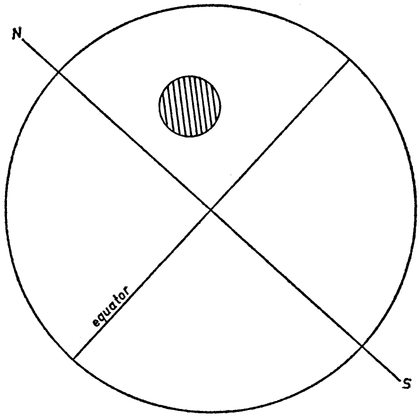


Fig. 2.7

3) Finally the Earth is supposed to lie neither on the axis, nor with the same distance from the poles (Figure 2.7). This assumption leads to all the objections raised against the two former positions, and is clearly impossible.

But Ptolemy has still another reason for the central position of the Earth, taken from eclipses of the Moon. It is a matter of experience that such eclipses occur only at oppositions (or Full Moons) when the Sun, Earth and Moon are on a straight line connecting diametrically opposite points of the heavens. But if the Earth were not central it could happen that it came between the Sun and the Moon at positions which were not diametrically opposite but had a difference of longitude smaller than

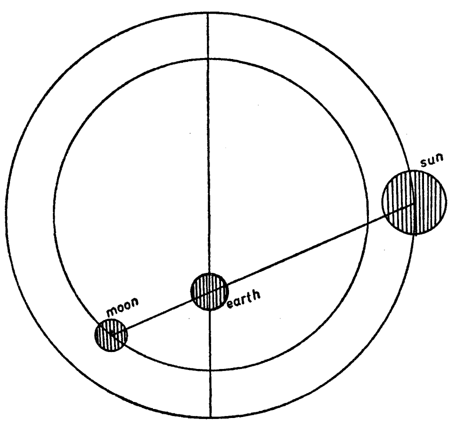


Fig. 2.8

180°. This argument is only hinted at, but Theon has developed it more fully (ed. Rome, p. 78ff.). It rests on the assumption that the Earth is outside the centre of the universe, but that both Sun and Moon are moving upon circles around this centre (Figure 2.8). In that case it is obvious that in the eclipse shown in the figure the line through Sun, Earth, and Moon will not divide the heavens into equal parts.

The Size of the Earth

Long before Ptolemy's time the absolute size of the Earth had been determined to a fair degree of accuracy by Eratosthenes (about 275–194 B.C.) and Posidonius (about 135–50 B.C.). Their results are not mentioned in the Almagest. Perhaps Ptolemy was unaware of them, however difficult to believe that may be. But the account of Eratosthenes' method and result is known from the *De motu circulari corporum caelestium* by the astronomer Cleomedes (2nd century A.D.), and it seems that this book was unknown to Ptolemy when he wrote the Almagest. The reason is that he never mentions Cleomedes as the discoverer of atmospheric refraction. This astronomically very important phenomenon is mentioned in the Almagest [I, 3; Hei 1, 13], but only as explanation of the changing apparent size of the heavenly bodies near the horizon. In his *Optics* (V, 23–30) Ptolemy examines astronomical refraction in more detail, but this work was completed at a later date. In any case, only relative measures of Earth, Sun, and Moon are given in the Almagest [V, 16; Hei 1, 426] with the radius of the Earth as the preferred astronomical unit (cf. below p. 213).

In the present chapter [I, 6; Hei 1, 20] Ptolemy is concerned only with the size of the Earth compared with the distance of the fixed stars. Since the latter are supposed to belong to a sphere (the firmament) concentric with the Earth, the problem is to find the ratio between the radius of the Earth and that of the starry sphere. Here Ptolemy argues that the Earth must be likened to a mere point compared with the firmament. The main reason is that the fixed stars keep their mutual distances and constellations unchanged no matter where the observer places himself on the Earth. In other words, the fixed stars are too far away to show any daily parallax. This word is not used here, but introduced later in the Almagest [V, 11; Hei 1, 401].

The insignificant size of the Earth appears also from the fact that any horizontal plane (i.e. any tangential plane to the surface of the Earth) is seen to divide the heavenly sphere into two exactly equal parts. If the Earth had an appreciable size only a plane through its centre could do that.

Particularly impressive is the fact that the point of a gnomon, or the centre of an armillary sphere, represents the centre of the Earth equally well wherever they are placed on its surface. (This means that stellar coordinates measured with such instruments may be regarded as geocentric, whereas the Moon is so near that it is necessary to reduce observed apparent positions to true geocentric positions by means of a particular theory of parallaxes).

The absence of daily parallax in the fixed stars would have set Ptolemy free to adopt

the hypothesis of the rotating Earth as an explanation of the diurnal motion of the heavens. Although this hypothesis was known to Greek astronomers, for instance, Hicetas, Ecphantus, Philolaus, Heracleides, and Aristarchus, Ptolemy refers to it in the Almagest only to refute it. This refutation is the purpose of the following chapter.

The Immobility of the Earth

That the Earth remains immovable at the centre of the universe is first proved very briefly by an astronomical argument [I, 7; Hei 1, 21]. If the Earth moved away from the centre it would necessarily come into one or another of the impossible positions already refuted [I, 5]. If Ptolemy was aware that this argument is questionable he does not tell us so. In fact, having just proved that the Earth is like a point compared with the firmament, Ptolemy would have to admit that a displacement of the Earth equal to, say, its own diameter would not influence the observed phenomena. Perhaps he would maintain that it would affect the relative position of Earth, Moon, and Sun and thus be impossible for the reason based on the phenomena of eclipses. In any case the proof needs further elucidation which is omitted, Ptolemy hastening on to a series of proofs based on physical considerations and comparable to those given by Aristotle who, by the way, dismisses the astronomical proof as quickly as Ptolemy (*De caelo* ii, 14, 296 a).

The first of the physical proofs is based on the Aristotelian theory of gravity. Ptolemy seems to regard gravitation very much like Aristotle as a natural tendency of heavy bodies to move towards the centre of the world as their 'natural place' of rest. Since we know that the Earth is placed in the middle of the universe, it follows that all heavy bodies moving freely under gravity must strike the surface of the Earth. This happens always in a vertical direction perpendicular to the tangential plane of the Earth at the point of impact. Since the vertical is conceived here as the direction towards the centre of the world, and since a sphere concentric with the universe is the only surface with a tangential plane everywhere perpendicular to the vertical, we have here an argument for the Earth being spherical and concentric with the universe. But this is no more than what we know already, and no proof of the immobility of the Earth, which might still have a diurnal motion around the centre. It would have been necessary to underline the principle that heavy bodies move towards the centre in order to be at rest there, but this is not stressed by Ptolemy.

Ptolemy proceeds with a refutation of those who cannot imagine so great and heavy a body as the Earth to be at rest without its inclining to one side or another. Aristotle had refuted this idea by means of his doctrine of natural motion: no substance has more than one natural motion (*De caelo* i, 2, 269 a), and since that of earth and other heavy particles is towards the centre of the world, it follows that the Earth as a whole cannot have a natural tendency to move sideways. It is interesting that here Ptolemy has a quite different argument based on the physical notion of pressure. As already mentioned, all matter is supposed to consist of molecules

(page 37). Now the Earth is extremely small compared with the celestial spheres. This means that it is subjected to an immense pressure by the ether molecules. Since these are all alike and surround the Earth evenly on all sides, the pressure will act uniformly everywhere on the Earth, and keep it in place at the centre, the pressure at one point of the surface being equal and opposite to that at the opposite point.

The argument is not quite clear, since Ptolemy on the one hand operates with the notion of ethereal molecules exerting a pressure towards the centre, while on the other hand he acknowledges that incorruptible matter has a natural, circular motion around the centre. Furthermore, he tries also to take into account the natural down- and upward motions of the heavy and light molecules in the neighbourhood of the Earth, which is a further complication. But in any case, it is obvious that Ptolemy's belief in the immobility of the Earth rests upon a line of thought of a more physical character (in the modern sense of the word) than the Aristotelian doctrine of natural places and natural motions, although he has not succeeded in liberating himself from the latter[7]).

Having established his own position, Ptolemy devotes the remaining part of this chapter to an attack on 'certain people' who assumed a diurnal rotation of the Earth around its own axis. At the beginning Ptolemy admits that from a purely astronomical point of view nothing speaks against the hypothesis of a rotating Earth, which is even simpler than that of a celestial universe rotating about an immobile Earth. Theon (ed. Rome, p. 89 ff.) develops this point, showing in great detail that all the diurnal phenomena are the same in either case. This means that Aristarchus's hypothesis can only be refuted by 'physical' arguments.

These arguments are based on the assumption that the rotation of the Earth would create atmospheric phenomena contrary to experience; for instance, that clouds would be overtaken by the Earth and that we should always have a strong wind towards the East. Ptolemy is aware that the supporters of the hypothesis tried to avoid this difficulty by the assumption that the atmosphere takes part in the rotation with the same velocity; but that is refuted for reasons taken from the doctrine of the free fall of bodies. If a heavy body is thrown upwards along the vertical it is supposed to return to the Earth along the same geometrical line, thus striking the surface at a point different from that from which it started. This does not agree with everyday experience. Ptolemy is aware, too, that thrown or flying bodies could be, as it were, united with the air. In that case they would not be overtaken by the Earth but would remain at the same place forever, contrary to what we observe in projectiles and birds. It has to be admitted that this is a very weak argument, indicating that Ptolemy has not gone deeply into the Aristotelian doctrine of violent motion.

7) That Ptolemy is on the move away from a purely Aristotelian doctrine is seen also from his (now lost) treatise *On the balance*. According to Simplicius he maintained here that neither water nor air has any weight at its natural place, since e.g. divers never feel the weight of the water above them, even at great depth. – Simplicius: *Comm. in de Caelo* IV, 4, cf. Thomas: *Greek Mathematical Works*, II, 411. It would have been interesting to see how such an anti-Aristotelian doctrine was dealt with in Ptolemy's other lost work *On the Elements*.

The Two Principles of the Motion of the Heavens

Until now we have only discussed phenomena connected with the daily motion of the heavens around the axis through the poles. This motion causes all the stars to move on circles parallel to the equator. Ptolemy explains the etymology of this name by the fact that the equator always and everywhere is divided into two equal parts by the horizon just as day and night are equal at the dates when the Sun is on the equator [I, 8; Hei 1, 26].

In the present chapter he explains the reasons why this first motion of the heavens is not the only one. In fact while the majority of the stars always rise and set at the same points of the horizon, there are a few wandering stars which behave otherwise. This is the case with the Sun, the Moon, and the five planets. Further observations show that these seven bodies perform individual motions towards the East relative to the fixed stars and at the same time they take part in the diurnal rotation. It looks as if they are moving on a circle inclined to the equator. This is, in fact, exactly the case for the Sun, whose apparent path among the fixed stars is a great circle, called the ecliptic, and intersecting the equator at two diametrically opposed points called the vernal and the autumnal equinox. At the former the Sun crosses the equator in the direction from South to North and at the latter in the opposite way.

The six remaining planets perform an even more complicated motion. Roughly speaking, they follow the ecliptic, completing their revolutions in slightly variable periods. But at the same time they have a motion in latitude, i.e. they depart a little from the ecliptic. In the first approximation it would seem that they revolve on great circles forming a small angle with the ecliptic and cutting it at two opposite points called the ascending and the descending node.

Conclusion

What has been said until now comprises everything that Ptolemy deemed necessary for the understanding of the picture of the universe upon which the rest of the Almagest is based. To the history of astronomy these first chapters are interesting not only because of what Ptolemy says, but because of the many points he is passing over in silence. Only one important question must be mentioned here. It is often said that the Almagest deals with theoretical astronomy in a purely mathematical and formalistic way, as if Ptolemy were uninterested in the question of the physical relevance of his geometrical models. But, as we have seen, there is a picture of the physical structure of the universe behind the Almagest. Thus the fixed stars are supposed to exist in a particular sphere concentric with the Earth. As for the planets, Ptolemy usually describes their motions by means of the purely mathematical concept of a circle, referring but rarely to material spheres, e.g. in the theory of the Moon [IV, 6; Hei 1, 301]. But this should not make us think that he belongs to what has been called the mathematical school of astronomers. We know from his Planetary Hypo-

theses that the doctrine of spheres was, in fact, an essential part of his astronomy and that Ptolemy is one of the sources of the many Mediaeval speculations on how the geometrical models could be transformed into a machinery of spheres and how the theories of the individual planets could possibly be compatible from this physical point of view (see p. 393 ff.). On the other hand, one is grateful that he did leave this physical point of view out of consideration in the Almagest. Had he chosen a different course, the work would have lost in clarity, and a lot of 'physical niceties' would have obscured the magnificent mathematical structure which is Ptolemy's lasting claim to fame as one of the greatest theoretical astronomers of all time.

CHAPTER 3

Ptolemy as a Mathematician

The Foundation of Geometry

Ptolemy's main scientific interest was the mathematical description of the phenomena of nature. In pure mathematics he did but little work, and what he wrote had but little value. Thus Simplicius tells us that *the gifted Ptolemy in his book* On Dimension *showed that there are not more than three dimensions; for dimensions must be determinate, and determinate dimensions are along perpendicular straight lines, and it is not possible to find more than three straight lines at right angles one to another, two of them determining a plane and the third measuring depth; therefore, if any other were added after the third dimension, it would be completely unmeasurable and undetermined* (*In libr. de caelo* i, 1; trans. Thomas ii, 411).

In another treatise *On the Parallel-Postulate* Ptolemy tried to give a proof of the 5th axiom in the Elements of Euclid. This famous axiom states that *if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles* (transl. Heath i, 202).

That this axiom is independent of Euclid's other presuppositions was not realized until non-Euclidean geometries emerged in the 18th and 19th centuries, and in the meantime numerous attempts were made to prove it from the other axioms. It can be formulated in several other ways than that found in Euclid, e.g. as the postulate that if two parallel lines are cut by a transversal line, the sum of the interior angles at the same side is equal to 180°. This is what Ptolemy tries to prove. Although his book is lost his argument has survived in Proclus (transl. Ver Eecke p. 168). A modern exposition is found in Heath (1921, ii, 295 ff.). The main argument is as follows: let the straight lines AB and CD be parallel. The transversal FG forms the interior angles BFG and FGD, and we suppose at first that

$$\text{angle AFG} + \text{angle CGF} > 180° \tag{3.1}$$

It follows that

$$\text{angle BFG} + \text{angle DGF} < 180° \tag{3.2}$$

However, Ptolemy argues that the fact that the lines AF and CG are 'no more parallel' than FB and GD implies that (3.1) leads to the inequality

$$\text{angle BFG} + \text{angle DGF} > 180° \tag{3.3}$$

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_3,

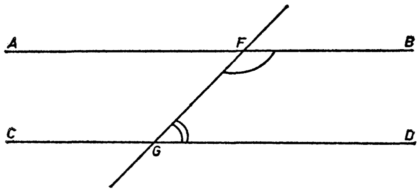


Fig. 3.1

Thus (3.2) is contradicted and the assumption (3.1) must be false. We shall not examine the remaining part of the argument since it is obvious that Ptolemy presupposed the very axiom which he meant to prove; the statement that AF and CG are 'no more parallel' than FB and GD amounts to the same as maintaining that only one parallel can be drawn to a given straight line through a point outside it – and it is well known that this latter statement is equivalent to the 5th axiom in Euclid.

Thus neither of Ptolemy's attempts to clear up some of the foundations of Euclidean geometry was crowned with success, and it may be no great loss to mathematics that his two books have perished.

The Mathematics of the Almagest

Fortunately for Ptolemy's reputation as a mathematician a much more impressive picture emerges from the mathematics set forth or implied in the Almagest. In fact, Ptolemy's main astronomical work is not only a highly technical treatise on astronomy, but has also a place of its own in the history of mathematics. One reason is that like many other great authors of advanced scientific works, Ptolemy had to provide his book with several mathematical sections explaining theorems and methods not found in the usual handbooks accessible to his readers. Thus he everywhere presupposes a thorough knowledge of the *Elements* of Euclid, which is referred to in several places without further comment. But as soon as he utilizes more special results by mathematicians like Apollonius, Hipparchus, or Menelaus, he is careful to give an exposition of what they achieved on the particular point in question. In this way the Almagest appears as an important source of the history of Greek mathematics between Euclid and Ptolemy's own times. Several important contributions would have been lost had they not been preserved in this way. Of course this is true to an even greater degree with respect to Ptolemy's own achievements in plane and spherical trigonometry. Actually, the Almagest contains the first surviving, consistent exposition of both branches of trigonometry and is, for that reason alone, a mathematical work of prime importance.

But this is not all. As we are going to see, Ptolemy's trigonometry is the science of how to make numerical calculations of plane and spherical triangles, and here we meet with a rather puzzling situation: while Ptolemy gives a very careful exposition of the geometrical foundation of trigonometry, he gives no explicit account of its

numerical aspects. He is silent about how calculations are carried out, in so far as he presupposes the whole technique of calculation without any attempt at stating general rules or methods; yet the Almagest abounds in calculations of an often very intricate nature. This gives the work a particular importance. On the one hand it denies the statement, commonly found in historical expositions, that Greek mathematics consciously neglected numerical methods in favour of purely geometrical procedures. On the other hand it confirms the general impression that for some reason or other Greek mathematicians dealt only reluctantly with such questions in the form of formal, explicit treatises. It is a fact that logistics, that is the practice of numerical methods, left only few and insufficient marks on Greek mathematical literature; but for that very reason the many computations of the Almagest become an important source of the history of early mathematics[1]).

One final mathematical characteristic of the Almagest is even more interesting. It is, of course, true that the notion of a mathematical function as such (i.e. in the classical sense) was unknown to the Ancients, and only slowly emerged at a much later time which it is difficult to determine precisely. However, both spherical astronomy and planetary theory abound in mathematical relations which are best described as functions of one, two, or three variables. Such relations a mathematical astronomer like Ptolemy had to deal with, and one of the most fascinating but least investigated mathematical aspects of the Almagest is the rudimentary 'theory of function' which can be extracted from its pages.

In the remaining part of this chapter we shall try to give an impression of all the points on which the Almagest is of interest to the history of mathematics. A complete exposition exhausting the subject is out of the question; but at least a glimpse of Ptolemy's true claim to be remembered as a mathematician will be substantiated in the following.

Ptolemy's Numerical Calculations

Since most of the mathematics of the Almagest consists of trigonometrical calculations in which geometrical entities are represented by numbers, we shall begin with a brief survey of how Ptolemy made numerical computations. This implies a certain knowledge of the number systems he employed. Here we can distinguish at least three different systems, of Greek, Egyptian, and Babylonian origin respectively.

When Ptolemy wants to write an integer he usually applies the Greek number system (see Thomas, i, 41–63). The Greeks had a decimal, additive system in which the letters of the alphabet, provided with a bar, served as symbols for the following 27 numbers: 1, 2 .... 9; 10, 20 .... 90; 100, 200 .... 900. Thus $\bar{\delta}$ means 4, $\overline{\iota\delta} = 14$, and $\overline{\rho\iota\delta} = 114$. Larger numbers were denoted by letters with bars and strokes; for

1) The principal works on the history of Greek logistics are the papers by Klein (1934) and Vogel (1936). The importance of astronomical calculations and tables for the development of logistics is hinted at by Wussing, 1965, p. 197.

example $,\bar{\alpha} = 1000$ and $,\bar{\beta} = 2000$. The sign M was used for 10,000 = one myriad. The system is rather clumsy, particularly because there is no limit to the number of separate symbols which might be necessary when large numbers are involved.

For fractions the Greeks had several more or less suitable systems. One of these used a numerator and a denominator, very much as we do today, except that their relative positions were not fixed, the numerator being sometimes written below the denominator. But the Almagest reveals the existence also of a much more primitive system. Here the letters of the alphabet were used once more, but this time with a dash instead of a bar, as in the following examples

$$\delta' = 1/4 \qquad \varepsilon' = 1/5 \qquad \kappa'\gamma' = 1/23$$

There were particular signs for 1/2 = 𐅵 and 2/3 = 𐅷. It is clear that only unit fractions (submultiples) could be expressed in this way, apart from 2/3. A very similar system is known from Egyptian mathematical texts as early as circa 2,000 B.C., where we also witness the old Egyptian method of resolving general fractions into sums of different unit fractions. This method survived in Greek mathematics and is known from e.g. a number of Greek papyri dating from Ptolemy's time and found in Egypt (Neugebauer, 1934, p. 137 ff.). Thus papyrus Michigan N° 145 gives a table (cf. Thomas, i, 46) of twenty-thirds equivalent to

$$\begin{aligned}
1/23 &= 1/23 \\
2/23 &= 1/12 + 1/276 \\
3/23 &= 1/10 + 1/46 + 1/115 \\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

Such representations are frequently found in the Almagest. For example Ptolemy says that according to Hipparchus the 5th century Athenian astronomers Meton and Euctemon found the length of the year to be 365 + 1/4 + 1/76 days [III, 1; Hei 1, 207]. In a similar way the time of a lunar eclipse is given as 1/2 + 1/3 hour before midnight [IV, 6; Hei 1, 303], and a time interval may be given as 354 days and 2 + 1/2 + 1/15 hours [IV, 6; Hei 1, 305].

The third system of numbers used in the Almagest is derived from the sexagesimal system used by Babylonian mathematicians and astronomers for more than 2,000 years before Ptolemy's time (Neugebauer, 1934, 93 f.). It differed from the other systems in various important ways. First, it used only two different symbols of which one (a vertical wedge) can be interpreted as 1 and the other (a horizontal wedge) as 10. By means of these symbols representations of the numbers 1, 2 .... 59 were formed in an additive way; for instance, 43 was written with 4 horizontal wedges followed by three vertical ones. Second, the system was positional in so far as the vertical wedge could be interpreted also as 60, or 60^2, or 60^{-1}, or any positive or negative integer power of 60. In other words, the value of any additive group of symbols depends on the position of the group in the series of 'places' by which the whole number is structured.

To take an example, the four groups 1 24 51 10 may be interpreted in many different ways, e.g.

$$1 \cdot 60^3 + 24 \cdot 60^2 + 51 \cdot 60 + 10$$

or

$$1 \cdot 60 + 24 + 51 \cdot 60^{-1} + 10 \cdot 60^{-2}$$

or, in general,

$$1 \cdot 60^n + 24 \cdot 60^{n-1} + 51 \cdot 60^{n-2} + 10 \cdot 60^{n-1}$$

In Babylonian mathematics the value of the integer n would appear from the context only. To avoid ambiguities in what follows we shall adopt the following convention introduced by O. Neugebauer: the integer part of the whole number is separated from the fractional part by a semicolon; other sexagesimal places are separated by commas. Thus

$$1;24,51,10$$

has the definite meaning

$$1 + 24 \cdot 60^{-1} + 51 \cdot 60^{-2} + 10 \cdot 60^{-3}$$

which is, by the way, the Babylonian value of the square root of 2 (see below p. 57 note 5).

Compared with the additive Greek system the Babylonian positional system has two obvious advantages: first, the number of necessary symbols is reduced to no more than 2, regardless of how large numbers we want to write, and second, there are no particular problems connected with the notation of fractions.

On the other hand the system has the disadvantage that not all fractions could be finite sexagesimal expressions. In fact, this was possible only in 'regular' fractions whose denominators could be written as $2^m \cdot 3^m \cdot 5^p$ where m, n, and p are integers, whereas a simple fraction like 1/7 or 1/11 would lead to an infinite sexagesimal expansion, which had to be broken off somewhere. This led, therefore, to rounded-off values and approximative computations which were just as unavoidable in astronomy as they were abhorrent to ordinary mathematical thought for the Greek geometers.

From our present knowledge of Greek mathematics it is impossible to decide who it was who first introduced the Babylonian system among the Greeks. But since Ptolemy states that Hipparchus found the length of the synodic month to be 29^d; 31, 50, 8, 20 it is certain that his great predecessor in the 2nd century B.C. was familiar with the sexagesimal system. At an even earlier date the Alexandrian astronomer Hypsikles (about 200 B.C.) wrote a treatise *Anaphoricos* in which he applied linear, Babylonian methods to spherical astronomy and divided the ecliptic into 360°. This puts the introduction of the sexagesimal system back to the beginning of the second century B.C. In the Almagest it is used for almost all trigonometrical and astronomical calculations, but not in precisely the same form as in Babylonian astronomy. In fact, Ptolemy (or Hipparchus) modified it in several ways:

1) He discarded the Babylonian cuneiform script, using everywhere Greek letters as number symbols, with the values already allotted to them in Greek mathematics. This made his computations more easily understandable to Greek readers, but augmented the number of symbols required, thus destroying some of the beautiful simplicity of the system.

2) Ptolemy also retained the usual Greek notation for the integer part of the number, writing e.g. 84 1/3 as 84;20 where the Babylonians would write 1,24; 20.

3) Finally Ptolemy introduced the Greek letter o (omicron) to denote an 'empty' sexagesimal place. This was done after the manner of late Babylonian mathematicians (Neugebauer, 1951, p. 26) who used a particular symbol to mark such a place in order to avoid confusion between, say

3 27 10 = 3; 27, 10

and

3 27 10 = 3; 27, 0, 10

In practice, the Ptolemaic omicron had the same function as the zero in the later Hindu numerals.

Computational Technique

Nowhere does the Almagest reveal how elementary arithmetical calculations were carried out. Obviously Ptolemy assumed his readers to be well versed in his own particular form of the sexagesimal system. Perhaps this was so in his own times; but later in Antiquity and in the Middle Ages one felt the need for a particular introduction to practical computations[2]). This might well be the case for a modern reader too. We cannot here go into all the aspects of this problem, but shall only indicate a few methods of particular importance.

Being a positional system, the sexagesimal system is very similar to the decimal system which we all know to-day. Thus it is easy to see that addition and subtraction can be carried out in exactly the same way. This is the case for multiplication too; but since the base is 60 instead of 10, the multiplication table is too long to be learned by heart. We know that the Babylonians had special tables for multiplication (Neugebauer, 1934, p. 18 f.) and may safely assume that Ptolemy used similar tables.

It is a little more difficult to imagine how he carried out more complex operations like division, or extraction of square roots, both of which occur very frequently in the Almagest. Fortunately, in his commentary Theon has given a detailed example of both cases.

2) A Greek treatise called *Introduction to the Almagest* and extant in at least 24 MSS has been examined by Mogenet (1956) who ascribed it to the Byzantine mathematician Eutocius (ca. A.D. 500). It deals with the isoperimetric problem, and multiplication, division, and root extraction in the sexagesimal system. In the Middle Ages there were several introductions to the so-called *Algorismus de minutiis* which was taught as a separate course in the universities and dealt with both decimal and sexagesimal fractions.

Theon (ed. Rome, p. 461) first illustrates the method of dividing numbers by means of the example

$$1515°;20,15 : 25;12,10 = 60°;7,33$$

In other words, he solves the equation

$$1515°;20,15 = 25;12,10 \cdot \left\{x + \frac{y}{60} + \frac{z}{60^2}\right\} \tag{A}$$

where x, y, and z are integers, y and z being smaller than 60. The method can be explained as follows:

First we disregard y and z and perform the first division finding $x = 60°$ as the first quotient, for $61°$ is too big. In fact, we have

$$25;12,\ 10 \cdot 60° = 1512°;10$$

$$25;12,\ 10 \cdot 61° = 1537°;22,10$$

which is found by direct computation, perhaps aided by a multiplication table.

Putting $x = 60°$ in equation (A) we get

$$1515°;20,15 = 1512°;10 + 25;12,10\left\{\frac{y}{60} + \frac{z}{60^2}\right\}$$

or

$$3°;10,15 = 25;12,10\left\{\frac{y}{60} + \frac{z}{60^2}\right\}$$

Multiplication by 60 gives

$$190';15 = 25;12,10\left\{y + \frac{z}{60}\right\} \tag{B}$$

Now the process is repeated: we perform the second division and find $y = 7'$ as the second quotient. Equation (B) is thereby reduced to

$$13';49,50 = 25;12,10 \cdot \frac{z}{60}$$

Multiplying once more by 60 we get

$$829'';50 = 25;12,10 \cdot z \tag{C}$$

This gives the third quotient $z = 33''$ and the complete solution is, to two sexagesimal places, 60°;7,33. Theon draws attention to the approximate character of the result, but does not say whether it is too big or too small. Actually $z = 33''$ makes the right hand side of (C) equal to 831″;41,30 so that the result is too big. If we wanted to proceed with more sexagesimal places we would put $z = 32''$; but if we want the nearest two-place value then Theon's result is the most correct.

This procedure is explained by Theon in words. He does not say whether any kind of algorithm is used. Now it is obvious that the method is very similar to that used today in ordinary divisions of decimal fractions, so that a simple algorithm can be constructed along modern lines. For the first division it would look like the following scheme:

$$\begin{array}{lrrrl}
\underline{25;12,10\;|} & 1515^\circ; & 20, & 15 & |\;\underline{60^\circ} \\
60^\circ \cdot 25 = \ldots\ldots\ldots\ldots & 1500^\circ & & & \\
\cline{2-2}
 & 15^\circ & & & \\
60^\circ \cdot 0^\circ;12 = \ldots\ldots\ldots & 12^\circ & & & \\
\cline{2-2}
 & 3^\circ; & 20, & 15 & \\
60^\circ \cdot 0^\circ;0,10 = \ldots\ldots & 0^\circ; & 10, & 0 & \\
\cline{2-4}
 & 3^\circ; & 10, & 15 & \\
= & 190';15 & & &
\end{array}$$

Another example of division has been analysed by Rome (1926, p. 13).

From division we now turn to the extraction of square roots. This is an important operation since it has to be performed each time a line is determined by the theorem of Pythagoras. Here Theon (ed. Rome, p. 473) gives a general rule which reads as follows, in I. Thomas's translation (Thomas, i, 61):

If we seek the square root of any number, we take first the side of the nearest square number, double it, divide the product into the remainder reduced to minutes, and subtract the square of the quotient; proceeding in this way we reduce the remainder to seconds, divide it by twice the quotient in degrees and minutes, and we shall have the required approximation to the side of the square area.

What this means is explained by an example (ed. Rome, p. 469 f.) which shows in detail how the square root

$$\sqrt{4500} = 67;4,55$$

is found. Without going into details of the long proof (formulated in the usual terms of geometrical algebra as known from Euclid) we can explain the main steps of the procedure as follows.

If we want an approximate solution to two sexagesimal places we must solve the equation

$$\sqrt{a} = x + \frac{y}{60} + \frac{z}{60^2} \tag{D}$$

where x, y, and z are integers, y and z being smaller than 60. According to the rule we must first find the nearest square number, i.e. we must determine x so that

$$x < \sqrt{a} < x + 1$$

or

$$x^2 < a < x^2 + 2x + 1$$

This can be done by trial and error, or more easily by means of a table of squares. To find y we first discard z in (D) which gives the inequality

$$\sqrt{a} > x + \frac{y}{60}$$

where x is known, or

$$a > x^2 + \frac{2x}{60} \cdot y + \left(\frac{y}{60}\right)^2$$

where we discard the term in y^2 and solve with respect to y

$$y < \frac{(a - x^2) \cdot 60}{2x} \tag{E}$$

Here the numerator is mentioned in the general rule as the remainder reduced to minutes. Choosing for y the largest integer satisfying (E), we put

$$x + \frac{y}{60} = p$$

and get from (D)

$$\sqrt{a} = p + \frac{z}{60^2} \tag{F}$$

or

$$a = p^2 + \frac{2p}{60^2} \cdot z + \left(\frac{z}{60^2}\right)^2$$

Repeating the procedure from above we discard the term in z^2 and have accordingly

$$a > p^2 + \frac{2p}{60^2} \cdot z$$

or

$$z < \frac{(a - p^2) \cdot 60^2}{2p} \tag{G}$$

where the numerator is the remainder reduced to seconds. If we choose z as the largest integer satisfying (G) we now have

$$\sqrt{a} = x;y,z$$

correct to two sexagesimal places.

These examples are sufficient to show that calculations are not easily performed in the sexagesimal system. Now the Almagest contains the results of a very great amount of calculation, many of them implying a great number of operations. It is almost impossible to believe that Ptolemy could have found time for this enormous

work, particularly when we consider that there are surprisingly few computational errors, and that the results presumably were checked by repeated calculation, or other methods. Even allowing for the greater speed made possible by auxiliary tables, and by great familiarity with sexagesimal technique possessed by ancient astronomers, one cannot help making the assumption that Ptolemy must have had assistants helping him with routine work in the numerical field. We do not know, however, if such assistants were servants in his own pay, or if the library in Alexandria provided secretarial assistance to its great scholars.

Calculation of Chords

The geometrical part of the Almagest begins with a chapter [I, 10; Hei 1, 31] in which Ptolemy explains the construction of a table of chords in a circle[3]), to be used in the numerous trigonometrical calculations in the rest of the work. It is almost certain that similar tables had been in use long before Ptolemy. Thus his commentator Theon of Alexandria tells us that *an investigation of chords in a circle is made by Hipparchus in 12 Books, and again by Menelaus in 6 Books* (ed. Rome p. 451). This is all the more probable as Hipparchus would have been unable to develop his theory of the Sun without a trigonometrical table[4]). But Ptolemy's table of chords is the first surviving specimen, and his account of its construction is the first treatise on trigonometry known to us. We shall therefore examine his method in some detail.

Like earlier Greek mathematicians (see page 51) Ptolemy adopts the Babylonian division of a circle into 360°. His object is to determine the length of the chord of an arc of x degrees in a circle of radius R. Since this length must be proportional to R it is sufficient to solve the problem for a standard circle. In modern trigonometry this standard circle usually has a radius equal to one unit of length; but in order to use the sexagesimal number system (see below p. 149) as far as possible Ptolemy gives his standard circle a radius of 60 units of length. In the following we shall denote the length of a chord corresponding to x degrees of arc in a circle with $R = 60^p$ by $ch(x°)$. The determination of $ch(x°)$ involves the following steps of arguments:

(1) A certain number of such chords can be found from simple properties of regular polygons. Ptolemy begins by considering the semicircle ABΓ with the centre Δ, in which the line ΔΓ is halved by E, and Z is determined by EZ = EB (Figure 3.2). From Euclid XIII, 9–10 we have

$$\Delta Z = ch(36°) \quad \text{and} \quad BZ = ch(72°)$$

This is well known to Greek mathematics from very early times; but what is new in Ptolemy is that he calculates these chords, expressing them in sexagesimal numbers as

$$ch(36°) = 37;4,55$$
$$ch(72°) = 70;32,3$$

3) See to the following Ideler (1812), Delambre (1817, ii, 36 ff.), Cantor (1892, 350 f.), Braunmühl (1900, i, 10–30) and Aaboe (1964, 101 ff.).
4) A reconstruction by G. J. Toomer of Hipparchus' trigonometrical table is to be published in *Centaurus*, vol. 18.

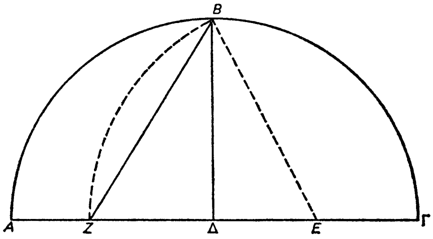


Fig. 3.2

Another well known property of the circle gives

$$\mathrm{ch}(60°) = 60;0$$

while the side of the inscribed square is

$$\mathrm{ch}(90°) = 84;51,10$$[5]

(2) More chords can be found by means of the relation

$$\mathrm{ch}(180° - \alpha) = 120^2 - \mathrm{ch}(\alpha)^2 \tag{3.4}$$

which follows from the theorem of Pythagoras applied to the triangle ABΓ (see Figure 3.3) in which B = 90°. In this way Ptolemy determines

$$\begin{aligned} \mathrm{ch}(108°) &= 97;\ 4,56 \\ \mathrm{ch}(120°) &= 103;55,23 \\ \mathrm{ch}(144°) &= 114;\ 7,37 \end{aligned}$$

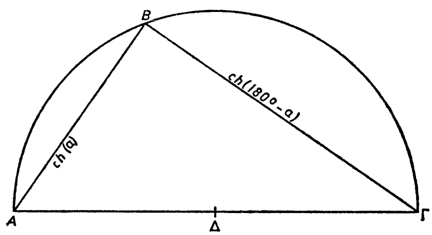


Fig. 3.3

Consequently we now have a short table of chords ch(n°) where n° is a member of the set

36°, 60°, 72°, 90°, 108°, 120°, 144°

(3) The next step is to prove a lemma enabling us to combine a chord with another line than the diameter. It has become known as the Theorem of Ptolemy. But,

5) Dividing this number by 60 (the radius of the circle) we find 1; 24,51,10 which is precisely equal to the value of $\sqrt{2}$ known from Babylonian mathematics, in particular from the tablet YBC 7289. See Neugebauer and Sachs, 1945, p. 42.

being of an elementary character, it probably dates from an earlier period and is only stated and proved for the first time in the Almagest [I, 10; Hei 1, 36]. The proof is as follows (cf. Figure 3.4).

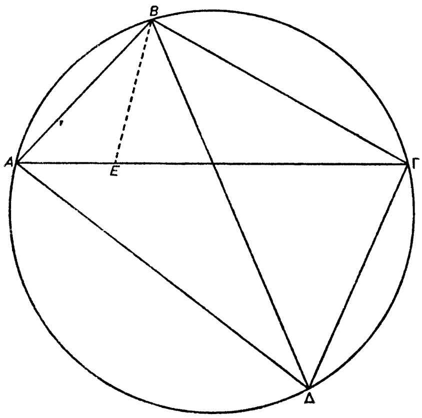


Fig. 3.4

In a circle is inscribed a quadrilateral ΑΒΓΔ, and the point E is determined on ΑΓ so that angle ABE = angle ΓΒΔ. By means of the similar triangles ABE $\sim$ ΔΒΓ we have

$$AB \cdot \Delta\Gamma = AE \cdot B\Delta$$

In the same way, the similar triangles

$$AB\Delta \sim EB\Gamma$$

give

$$A\Delta \cdot B\Gamma = E\Gamma \cdot B\Delta$$

Adding these relations we obtain the theorem: In an inscribed quadrilateral the product of the diagonals is equal to the sum of the two products of opposite sides, or

$$AB \cdot \Gamma\Delta + A\Delta \cdot B\Gamma = A\Gamma \cdot B\Delta \tag{3.5}$$

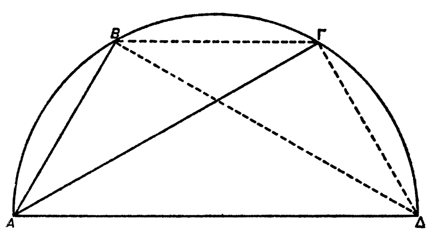


Fig. 3.5

(4) Let us now consider Figure 3.5 in which two known chords ch(α) = AB and

$ch(\beta) = A\Gamma$ are drawn from A in a semi-circle with the diameter $A\Delta = 120^p$. Since we have

$$B\Delta = ch(180° - \alpha)$$
$$\Gamma\Delta = ch(180° - \beta)$$

and

$$B\Gamma = ch(\beta - \alpha)$$

we find by means of (3.5) that

$$ch(\beta - \alpha) = \frac{ch(\beta) \cdot ch(180° - \alpha) - ch(\alpha) \cdot ch(180° - \beta)}{120} \tag{3.6}$$

Thus Ptolemy's theorem enables us to find the chord to an arc equal to the difference between two arcs the chords of which are known. For example, from the values of ch(90°) and ch(72°) already found we can calculate ch(18°). Combining ch(72°) with ch(60°) we get ch(12°), and from ch(18°) and ch(12°) we find ch(6°). It is seen that in this way we can compute all chords of the form $ch(n \cdot 6°)$, where n is an integer. In other words we can construct a table of chords with intervals of 6°.

It is interesting to see how (3.6) looks in modern notation in which a chord $ch(\alpha)$ is given by the formula

$$ch(\alpha) = 2R \cdot \sin\frac{\alpha}{2} \tag{3.7}$$

Putting $\beta = 2x$ and $\alpha = 2y$ we can write (3.6) in the form

$$\sin(x - y) = \sin x \cdot \sin(90° - y) - \sin(90° - x) \cdot \sin y$$

or

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \tag{3.8}$$

The corresponding formula

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{3.9}$$

can be deduced in a similar way, applying Ptolemy's theorem to the computation of a chord $ch(\beta + \alpha)$ where $ch(\alpha)$ and $ch(\beta)$ are given. This is actually done by Ptolemy later in the same chapter by a procedure analogous to that leading to (3.6). According to Tropfke (1928 p. 436) there is a tradition (mentioned by the 10th century Persian astronomer al-Bīrūnī) that already Archimedes possessed methods for computing chords of the sum or the difference of two given arcs (cf. Schoy 1925).

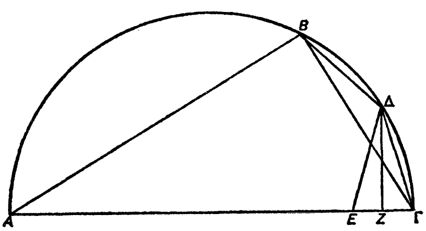


Fig. 3.6

(5) Now follows another lemma by means of which we can find the chord of the half of a given arc when the chord of the whole arc is known. In Figure 3.6 the chord $\mathrm{B\Gamma} = \mathrm{ch}(\alpha)$ is given and we seek $\Gamma\Delta = \mathrm{ch}(\alpha/2)$ when Δ is the middle point of the arc $\mathrm{B\Gamma}$.

If E is determined on $\mathrm{A\Gamma}$ so that

$$\mathrm{AE} = \mathrm{AB}$$

the triangles $\mathrm{AB\Delta}$ and $\mathrm{AE\Delta}$ will be congruent, wherefore

$$\mathrm{E\Delta} = \Delta\mathrm{B} = \Delta\Gamma$$

Thus the perpendicular $\Delta\mathrm{Z}$ will make $\mathrm{EZ} = \mathrm{Z\Gamma}$, from which follows

$$\mathrm{Z\Gamma} = \tfrac{1}{2}(\mathrm{A\Gamma} - \mathrm{AB})$$

Since the angle $\mathrm{A\Delta\Gamma} = 90°$ it follows from Euclid VI,8 that

$$\Gamma\Delta^2 = \mathrm{A\Gamma} \cdot \tfrac{1}{2}(\mathrm{A\Gamma} - \mathrm{AB})$$

or

$$\Gamma\Delta = \sqrt{60(120 - \mathrm{AB})} \tag{3.10}$$

which, in our notation, means

$$\mathrm{ch}(\alpha/2) = \sqrt{60[120 - \mathrm{ch}(180° - \alpha)]} \tag{3.11}$$

It is easily seen that this corresponds to the relation

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{2}} \tag{3.12}$$

It seems that this formula, or rather the procedure leading to it, was known before Ptolemy. Special cases are found in Archimedes who, in *The Measurement of the Circle* (transl. Heath p. 93 f.), computes what we would call sin 15° from sin 30° (Tropfke 1928, p. 441 f.).

It goes without saying that, by means of (3.12) and the chords already found, we are now able to construct a table of chords with intervals of 3°. Furthermore, from ch(3°) we find

$$\mathrm{ch}(3/2°) = 1;34,15$$

and

$$\mathrm{ch}(3/4°) = 0;47,\ 8$$

Thus it would be possible to take one further step and reduce the interval to 3/2°, or even 3/4°.

(6) But this was not enough, since Ptolemy wanted a table of chords with intervals of 1/2°. Such a table could be constructed by means of the previous formulas if only we were able to determine ch(1/2°). Since ch(3/2°) is known this would imply the trisection of an angle; but that was one of the problems which Greek mathematics

had never succeeded in solving by methods involving constructions with ruler and compasses only. Actually, we know today that with these restrictions it is insoluble[6]), except in particular cases, which do not include 3/2°.

This was realized by Ptolemy [I, 10; Hei 1, 42], who succeeded in avoiding the difficulty by means of the following lemma: let α and β be two arcs the chords of which satisfy the relation ch(α) > ch(β). It follows that

$$\frac{\mathrm{ch}(\alpha)}{\mathrm{ch}(\beta)} < \frac{\alpha}{\beta} \tag{3.13}$$

or

$$\frac{\mathrm{ch}(\alpha)}{\alpha} < \frac{\mathrm{ch}(\beta)}{\beta} \tag{3.14}$$

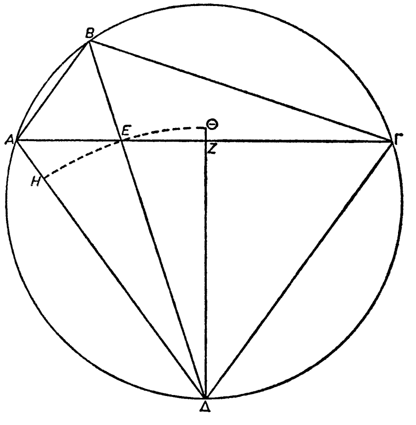


Fig. 3.7

This is proved geometrically by means of Figure 3.7.

Let the chords ΓB > BA be given, and let BΔ bisect the angle ABΓ. We then have

$$\Gamma\Delta = \Delta\mathrm{A}$$

and

$$\Gamma\mathrm{E} > \mathrm{EA}$$

The perpendicular to AΓ from Δ cuts AΓ at the middle point Z. A circle is drawn with radius ΔE and centre Δ intersecting AΔ at H and the prolongation of ΔZ at Θ. It is seen that

$$\text{sector } \Delta\mathrm{E}\Theta > \text{triangle } \Delta\mathrm{EZ}$$

and

$$\text{triangle } \Delta\mathrm{EA} > \text{sector } \Delta\mathrm{EH}$$

6) The various classical solutions based on other methods are given by Thomas, i, pp. 346–363.

from which follows that

$$\frac{\text{triangle } \Delta\text{EZ}}{\text{triangle } \Delta\text{EA}} < \frac{\text{sector } \Delta\text{E}\Theta}{\text{sector } \Delta\text{EH}}$$

This is, for obvious geometrical reasons, equivalent to

$$\frac{\text{ZE}}{\text{EA}} < \frac{\text{angle Z}\Delta\text{E}}{\text{angle E}\Delta\text{A}}$$

or

$$\frac{\text{ZE} + \text{EA}}{\text{EA}} < \frac{\text{angle Z}\Delta\text{E} + \text{angle E}\Delta\text{A}}{\text{angle E}\Delta\text{A}}$$

i.e.

$$\frac{\text{ZA}}{\text{EA}} < \frac{\text{angle Z}\Delta\text{A}}{\text{angle E}\Delta\text{A}}$$

Multiplication by 2 gives

$$\frac{\text{A}\Gamma}{\text{EA}} < \frac{\text{angle }\Gamma\Delta\text{A}}{\text{angle E}\Delta\text{A}}$$

from which follows that

$$\frac{\text{A}\Gamma - \text{EA}}{\text{EA}} < \frac{\text{angle }\Gamma\Delta\text{A} - \text{angle E}\Delta\text{A}}{\text{angle E}\Delta\text{A}}$$

or

$$\frac{\Gamma\text{E}}{\text{EA}} < \frac{\text{angle }\Gamma\Delta\text{E}}{\text{angle E}\Delta\text{A}}$$

Since BΔ bisects the angle at B, we have

$$\frac{\Gamma\text{E}}{\text{EA}} = \frac{\Gamma\text{B}}{\text{BA}}$$

On the other hand the ratio of the angles is equal to

$$\frac{\text{arc }\Gamma\text{B}}{\text{arc BA}}$$

We therefore have

$$\frac{\Gamma\text{B}}{\text{BA}} < \frac{\text{arc }\Gamma\text{B}}{\text{arc BA}} \qquad (3.15)$$

which was to be proved.

(7) This fundamental inequality permits us to confine a chord between to fixed limits, a procedure which seems to have been known already to Aristarchos in the 4th century B.C.[7]. In the Almagest it leads to the following relations

$$\frac{\text{ch}(3/4°)}{3/4} > \frac{\text{ch}(1°)}{1} > \frac{\text{ch}(3/2°)}{3/2} \quad \cdots\cdots \tag{3.16}$$

which, together with the known values of ch(3/4°) and ch(3/2°), gives

$$\text{ch}(1°) < \tfrac{4}{3}\cdot 0;47,\ 8 = 1;2,50$$

and

$$\text{ch}(1°) > \tfrac{2}{3}\cdot 1;34,15 = 1;2,50$$

Thus ch(1°) can differ from the value 1;2,50 by an amount which would appear only in the third sexagesimal place. Accordingly Ptolemy concludes that we have

$$\text{ch}(1°) = 1;2,50$$

By means of (3.12) we now find that

$$\text{ch}(1/2°) = 0;31,25$$

Consequently it is now possible for Ptolemy to construct a table with intervals of 1/2°. How this was achieved is explained only by a few examples. For instance, ch(2°) is found from ch(3/2°) and ch(1/2°), ch(2 1/2°) from ch(3°) and ch(1/2°), and so on.

The Table of Chords

The immediate result of the work outlined above is the great *Table of Chords* published by Ptolemy in the Almagest [I, 11; Hei 1, 48] as the first extant trigonometrical table in the history of mathematics. It represents a very considerable amount of numerical calculation and is the first of many similar tables in which Ptolemy used to compress a vast body of numerical material into a compact and handy form. The table has a total of 360 rows or lines arranged in 3 columns according to the following scheme.

arc α_n	chord ch(α_n)	sixtieths f(α_n)
1/2°	0^p;31,25	0^p;1,2,50
1°	1^p; 2,50	0^p;1,2,50
1 1/2°	1^p;34,15	0^p;1,2,50

7) Cf. Tropfke, 1928, p. 466 f. In fact, the inequality was used (without proof) by Aristarchus in his book *On the Sizes and Distances of the Sun and Moon* (see Heath 1913 p. 333), by Archimedes in *The Sand-Reckoner* (transl. Heath p. 226), and by Menelaus (see Bjørnbo, 1902, p. 112). But there is no evidence that it was proved until Ptolemy provided the proof given above.

The first column contains the arguments expressed as arcs α_n varying throughout the table from 1/2° to 180° with intervals of 1/2°. If n is the number of the row we have $\alpha_n = n \cdot 1/2°$.

The second column lists the chords $ch(\alpha_n)$ corresponding to the arguments of the same line of the first column, given in units of 1^p (see above p. 56) with two sexagesimal places. Ptolemy assures us that the values can be checked for errors in several ways, for instance by computing the chord to the difference of two arcs by (3.6), or to the supplementary arc by (3.4).

The third column is said to contain 'sixtieths', but actually it lists the function

$$f(\alpha_n) = \tfrac{1}{30}[ch(\alpha_n + 1/2°) - ch(\alpha_n)] \tag{3.17}$$

i.e. 1/30 of the difference between the chords N° (n + 1) and N° n. The values are computed to 3 sexagesimal places of the unit 1^p. This column is used for determining chords of arcs not listed in the first column. This is done by linear interpolation according to a procedure explained by a numerical example and corresponding to the formula

$$ch(\alpha) = ch(\alpha_n) + (\alpha - \alpha_n) \cdot f(\alpha_n) \tag{3.18}$$

where α_n is an argument of the first column determined by

$$\alpha_n < \alpha < \alpha_n + 1/2°$$

In modern notation the function $ch(\alpha)$ given by the table of chords is expressed by (3.7). If we apply the latter formula to twice the argument – i.e. to 2α – and solve with respect to $\sin\alpha$ we get

$$\sin\alpha = \frac{ch(2\alpha)}{2R} \tag{3.19}$$

This relation shows that the table of chords is equivalent to a modern table of sines in so far as we can find $\sin\alpha$ from $ch(2\alpha)$, i.e. by entering the table in the line corresponding to twice the angle of which we wish to find the sine. This doubling of the argument is expressed in a curious way in the Almagest, where we often find the statement that

angle α	$ch(2\alpha)$	$\sin\alpha$ derived from $ch(2\alpha)$ table	$\sin\alpha$ derived from modern table	diff. $\cdot 10^{-8}$
10°	20;50,16	0.17364814	0.17364817	−003
20°	41; 2,33	0.34202083	0.34202014	+069
30°	60; 0, 0	0.50000000	0.50000000	000
40°	77; 8, 5	0.64278240	0.64278760	−520
50°	91;55,32	0.76604629	0.76604444	+185
60°	103;55,23	0.86602546	0.86602540	+006
70°	112;45,48	0.93969444	0.93969267	+177
80°	118;10,37	0.98480787	0.98480775	+012

a given angle of α degrees is equal to 2α degrees *to 2 right angles* [e.g. II, 5; Hei 1, 99]. That the measure is doubled implies that the unit is halved, i.e. that Ptolemy doubles the argument by measuring it in half degrees. This terminology is used everywhere, and we shall not comment upon it in each particular case.

If we derive sine-values of the angles represented in the table of chords and compare them with modern values we can get an impression of how exact the table is. A few instances are listed above. The 8 values compared here show that the sines derived from the table of chords are all of them correct to the 5th decimal place (cf. Braunmühl 1900, i, p. 22).

Plane Trigonometry

In the *Almagest* all trigonometrical calculations are carried out by means of the table of chords. Since both cos α, tan α, and cot α can be expressed in terms of sin α, this table is, in fact, a sufficient basis for the treatment of any problem involving elementary trigonometrical functions. But it gives Ptolemy's methods a peculiar character and it is sometimes difficult to compare his procedures with modern trigonometrical methods[8]).

However, so much is clear that much of the trigonometry of the *Almagest* is based on relations between sides and angles of the right-angled triangle. In Ptolemy there is no consistent notation, but in the following we shall always denote the sides by a, b, c, and the corresponding opposite angles by A, B, C. In a right-angled triangle the right angle shall be C and the hypotenuse c. An examination of the *Almagest* shows that Ptolemy is able to perform calculations corresponding to the following relations:

$$a^2 + b^2 = c^2 \quad \text{(Euclid I, 47)} \tag{3.20}$$

$$\sin A = \frac{a}{c} \tag{3.21}$$

$$a = c \sin A \tag{3.22}$$

$$\sin(90^\circ - A) = \frac{b}{c} \qquad [= \cos A] \tag{3.23}$$

$$b = c \sin(90^\circ - A) \; [= c \cos A] \tag{3.24}$$

$$b = \frac{a \sin B}{\sin(90^\circ - B)} \qquad [= a \tan B] \tag{3.25}$$

Of course, no such formulae are found in the *Almagest*, in which there is, in general, very little mathematical formalism. Instead, Ptolemy usually gives a detailed account

8) For a brief survey of Ptolemy's trigonometry, see Czwalina (1927). A short, general account of the early history of both plane and spherical trigonometry with much emphasis on Muslim and Indian contributions has been given by Kennedy (1969). Luckey (1940) deals in greater detail with spherical trigonometry among the Arabs.

of the procedures applied in the actual calculations. From such procedures we can extract formulae in modern notation. But it should be remembered that when in the following we refer to, say, formula (3.23) we are really referring to the procedure used by Ptolemy for a calculation of which (3.23) is only a kind of summary in modern mathematical shorthand.

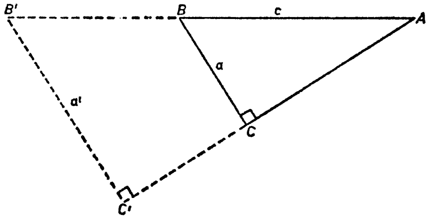


Fig. 3.8

As an example we shall consider the procedure leading to (3.21), corresponding to the problem: Given c and a, to find A. The solution is most easily understood if we assume that Ptolemy had in mind an auxiliary triangle AB'C', similar to ABC, but constructed so that $AB' = 120^p$ (see Figure 3.8). We then have

$$\frac{a'}{a} = \frac{120^p}{c} \quad \text{or} \quad a' = a \cdot \frac{120^p}{c}$$

We can then imagine a circle with the diameter AB' and passing through C'. In this circle the line $a' = B'C'$ will be chord to an arc $B'C' = 2A$ so that (3.19) gives

$$\sin A = \frac{a'}{120^p}$$

which, combined with the expression for a', gives (3.21). From (3.21) we are able to deduce (3.22) by a simple operation. Because of the lack of formal relations Ptolemy was unable to do so, and had to describe the inverse procedure of (3.21) in detail [see e.g. V, 7; Hei 1, 386–387].

The crux of the matter is that we always have to introduce a line of the length 120^p as hypotenuse of an auxiliary figure, because the table of chords presupposes a circle of that diameter. This is also clear from the procedure corresponding to (3.25), taken from the theory of shadows [II, 5; Hei 1, 99]. The problem is to find b when a and B are given (cf. Figure 3.9). As before, an auxiliary triangle is constructed with $BA' = 120^p$. From (3.22) we have

$$b' = A'C' = 120^p \cdot \sin B$$

and with $A = 90° - B$

$$a' = BC' = 120^p \cdot \sin (90° - B)$$

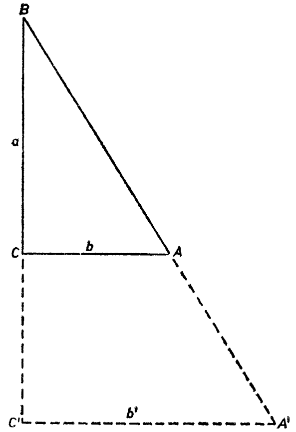


Fig. 3.9

Because of the similarity of the triangles we have

$$\frac{b}{a} = \frac{b'}{a'} = \frac{\sin B}{\sin (90° - B)}$$

which is equivalent to (3.25).

Thus Ptolemy is able to handle all the fundamental relations within right-angled triangles. Oblique-angled triangles are always decomposed into right-angled ones in a systematic way, so that here we can also distinguish well established procedures. A few typical examples are the following.

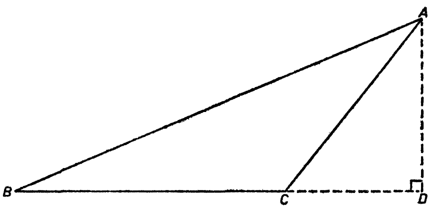


Fig. 3.10

Case A: Given a, b, and C, to find B [cf. III, 5; Hei 1, 247 f.]. First the point D is determined as shown in Figure 3.10 so that we now have two right-angled triangles. From triangle ACD we have

$$AD = b \sin (180° - C) = \quad b \sin C$$

and

$$CD = b \cos(180° - C) = -b \cos C$$

Applying (3.20) to triangle ABD we find

$$c = \sqrt{(b \sin C)^2 + (a - b \cos C)^2} \tag{3.26}$$

Then the angle is found by (3.21):

$$\sin B = \frac{AD}{c} = \frac{b \sin C}{\sqrt{(b \sin C)^2 + (a - b \cos C)^2}} \tag{3.27}$$

valid also if C is an acute angle. It is solved in the usual way by the Table of Chords. If Ptolemy had introduced the tangent function (3.27) could have been written as

$$\tan B = \frac{b \sin C}{a - b \cos C} \tag{3.27 a}$$

Case B: In another important case we have: Given a, b, and A, to find B [III, 7; Hei 1, 254]. The point D is constructed as before. From triangle ACD it follows that

$$CD = b \sin (180^\circ - A) = b \sin A$$

and from triangle BCD

$$\sin B = \frac{CD}{a} \tag{3.28}$$

Here Ptolemy is not far away from the modern sine relations. For combining the two relations, we find

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{3.28 a}$$

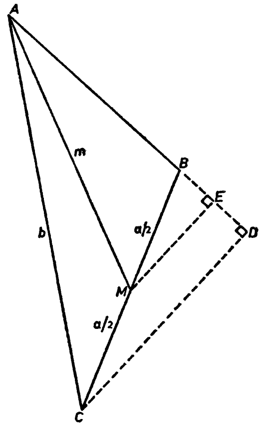


Fig. 3.11

A more complicated, but very important trigonometrical problem often found in planetary theory in the Almagest (see e.g. p. 279) is the determination of A from a, B, and m, where m is the median to a (Figure 3.11). This involves the following procedure: we first find D and E, the orthogonal projections of C and M on AB produced. This gives two rightangled triangles BCD and BME, from which we find

$$CD = 2 \cdot ME = \quad a \sin B$$

and

$$BD = 2 \cdot BE = -a \cos B$$

Applying (3.20) to triangle AME we have

$$\left(c - \frac{a}{2}\cos B\right)^2 = m^2 - \left(\frac{a}{2}\sin B\right)^2$$

or

$$c = \sqrt{m^2 - \left(\frac{a}{2}\sin B\right)^2} + \frac{a}{2}\cos B$$

This gives

$$AD = c + BD$$

so that we can find b by (3.20) applied to triangle ACD:

$$b = \left[\left(\sqrt{m^2 - \left(\frac{a}{2}\sin B\right)^2} - \frac{a}{2}\cos B\right)^2 + (a \sin B)^2\right]^{\frac{1}{2}} \tag{3.29}$$

Finally A is found as in case B; this is the usual procedure. Of course, we could have found A without knowing b by (3.27 a), from the relation

$$\tan A = \frac{CD}{AD} = \frac{a \sin B}{\sqrt{m^2 - \left(\frac{a}{2}\sin B\right)^2} - \frac{a}{2}\cos B} \tag{3.30}$$

but this procedure was not possible for Ptolemy.

The Theorem of Menelaus

The trigonometrical procedures mentioned above are used in the Almagest without any particular comments on the part of the author, who must have regarded his readers as sufficiently familiar with this kind of computation. But besides these procedures Ptolemy's plane trigonometry contains a number of theorems which are stated in the form of explicit propositions and carefully proved. They are not found in Euclid and may not have been universally known among Greek mathematicians, although they were not all of them new. The most important is the famous theorem named after Menelaus, even though, according to Bjørnbo (1902, p. 99), it may have been known already to Hipparchus (cf. also Rome 1933). Ptolemy does not mention its author, which is all the more remarkable considering that Menelaus is quoted in the Almagest [VII, 3; Hei 2, 30] as a geometer who in A.D. 98 made astronomical observations in Rome. This is about all we know of his life; his work on spherical problems will be mentioned in more detail below (p. 73).

Menelaus's theorem is proved by Ptolemy in the following way. In Figure 3.12 the lines AB and AΓ are given. Let Δ be a point on AB, and E a point on AΓ, and let BE and ΓΔ intersect each other at Z. Drawing EH $\neq$ ΓΔ we have the similar triangles

$$\Gamma\Delta A \sim EHA$$

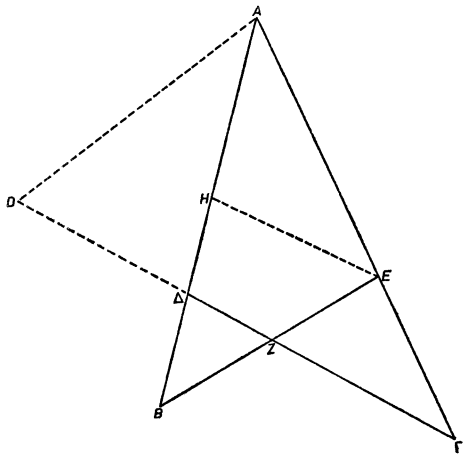


Fig. 3.12

from which it follows that

$$\frac{\Gamma A}{AE} = \frac{\Gamma \Delta}{EH}$$

The triangles BEH $\sim$ BZΔ are also similar, which gives

$$\frac{EB}{ZB} = \frac{EH}{\Delta Z}$$

By multiplication we have

$$\frac{\Gamma A}{AE} \cdot \frac{EB}{ZB} = \frac{\Gamma \Delta}{\Delta Z}$$

or

$$\frac{\Gamma A}{AE} = \frac{\Gamma \Delta}{\Delta Z} \cdot \frac{ZB}{BE} \tag{3.31}$$

which is the first form of the theorem proved by Ptolemy [I, 13; Hei 1, 69].

In a similar way the relation

$$\frac{\Gamma E}{EA} = \frac{\Gamma Z}{Z\Delta} \cdot \frac{\Delta B}{BA} \tag{3.32}$$

is proved by means of a line through A parallel to BE [ibid., Hei 1, 70]. Two other possible relations of the same type are not mentioned. The importance of these relations will appear in the following section on Ptolemy's spherical trigonometry.

Ptolemy concludes the plane section of his trigonometry by proving two lemmas (cf. Kennedy 1969, p. 343) of which the first [I, 13; Hei 1, 70] runs as follows (in modern notation).

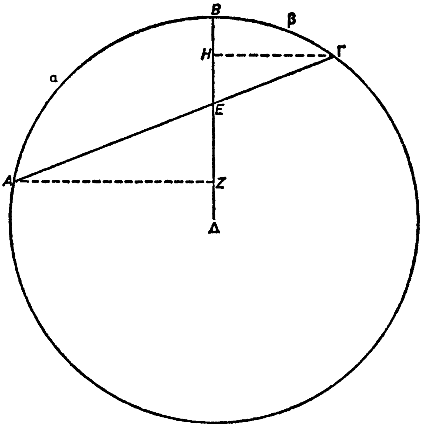


Fig. 3.13

Lemma 1: if $(\alpha + \beta)$ and $\dfrac{\sin\alpha}{\sin\beta}$ are given, then the arcs α and β can be found separately.

The proof rests on Figure 3.13 where $AB = \alpha$ and $B\Gamma = \beta$. The radius ΔB is intersected by the chord $A\Gamma$ in E, and A and Γ are projected upon ΔB in Z and H respectively. Since the two triangles in the figure are similar, we have

$$\frac{AZ}{\Gamma H} = \frac{AE}{E\Gamma}$$

But according to (3.7) we have

$$\frac{AZ}{\Gamma H} = \frac{R\sin\alpha}{R\sin\beta}$$

so that

$$\frac{\sin\alpha}{\sin\beta} = \frac{AE}{E\Gamma} \tag{3.33}$$

This entails a simple construction of the separate arcs α and β: the chord $A\Gamma$ of the given arc $(\alpha + \beta)$ is divided internally into the proportion $\dfrac{\sin\alpha}{\sin\beta}$ by the point E (Ptolemy explains in detail how this is done.) Then a radius through E cuts the circle at a point B dividing arc $A\Gamma$ in $AB = \alpha$ and $B\Gamma = \beta$.

Finally, Ptolemy proves [I, 13; Hei 1, 72]

Lemma 2: if $(\alpha - \beta)$ and $\dfrac{\sin\alpha}{\sin\beta}$ are given, the arc β can be found.

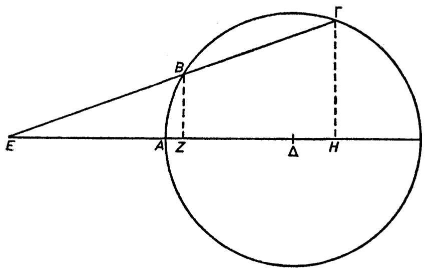


Fig. 3.14

This is proved by means of Figure 3.14, where we suppose

$$\text{arc } A\Gamma = \alpha \quad \text{and} \quad \text{arc } AB = \beta$$

ΓB and ΔA produced meet in E, and B and Γ are projected orthogonally on ΔA, Z and H being the projections of B and Γ respectively. Here we can easily prove the relation

$$\frac{\sin \alpha}{\sin \beta} = \frac{E\Gamma}{EB} \tag{3.34}$$

so that the point E divides BΓ externally in the ratio $\frac{\sin \alpha}{\sin \beta}$.

Then, if arc $B\Gamma = \alpha - \beta$ is given, we determine the point E which divides the chord BΓ externally in the given ratio $\frac{\sin \alpha}{\sin \beta}$. Then a line EΔ intersects the circle at a point A, so that arc $AB = \beta$.

Ptolemy does not discuss what happens if $\sin \alpha = \sin \beta$ (in which case AΔ becomes parallel to the chord BΓ, and $\beta = 180° - \alpha$); neither does he say that also arc $\alpha = AB\Gamma$ is determined by the construction. The two lemmas (3.33) and (3.34) are used for the proof of the spherical form of Menelaus' theorem (see below p. 73).

Spherical Trigonometry

Greek astronomers were interested in spherical problems at a very early date. In fact, the very first Greek mathematical texts which have come down to us complete in their original forms are two small treatises *On the Moving Sphere*, and *On Risings and Settings* by the astronomer Autolycus of Pitane (about 300 B.C.). The author was one of the first critics of the system of Eudoxus, and was thus interested in planetary theory, but his extant writings are concerned only with spherical astronomy. He shows that any point on a uniformly revolving sphere moves upon a circle parallel to the equator, and discusses the motion of various points relative to a fixed horizon. The two tracts became models for several later treatises of the same kind which in the course of time became known as *The Little Astronomy* (see Heath 1921, i, 348), in

contradistinction to the *Great Collection* (i.e. the Almagest). One of these works is the *Phaenomena* by Euclid, which is founded directly on Autolycus and extant in the original Greek. Another is the *Sphaerica* by Theodosius of Bithynia (circa 100 B.C.) who also wrote on spherical-astronomical subjects such as *On Days and Nights*, and *On Habitations*. The *Sphaerica* of Theodosius is conceived as a supplement to Euclid XII–XIII and modelled upon Euclid III (on the circle). Much of it deals with the usual circles of the heavenly sphere, but without direct reference to astronomy. There is no trigonometry as such in Theodosius, but a new result is a congruence theorem for spherical triangles (cf. Heath 1921, ii, p. 246 f.). There is no doubt that trigonometrical calculations in the strict sense were made by Hipparchos in the 2nd century B.C., but it is not until Menelaus that spherical trigonometry appears as a special branch of mathematics in another *Sphaerica*, of which several printed Latin versions were published by Maurolyco (1558), Mersenne (1644), and Halley (1758). They are all founded upon Arabic translations, the original Greek texts being lost.

The *Sphaerica* of Menelaus (see Bjørnbo 1902 and Rome 1933) is in three books, of which the first begins with a series of definitions; a spherical triangle is defined as a figure comprised between three great circles on the surface of a sphere. Then follow 35 propositions in which the congruence properties of such triangles are investigated. Prop. 11 states the sum of the angles to be greater than 180°. The second book contains some generalization of the theorems found in Theodosius (with different proofs), but still without spherical trigonometry, which only appears in the third book. Here the first proposition is Menelaus' theorem. The plane version of the latter has been considered above, and we shall now see how its spherical form is proved by Ptolemy[9]) in the Almagest [I, 13; Hei 1, 74], cf. Figure 3.15.

Let AB and AΓ be parts of two great circles upon which lie the points Δ and E respectively. The great circle through B and E cuts the great circle through Γ and Δ in

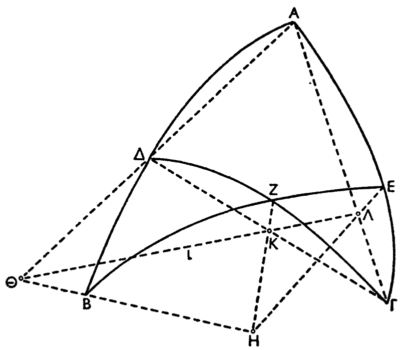


Fig. 3.15

9) Ptolemy's dependence on Menelaus appears from the fact that the proofs found in the Almagest are abbreviated versions of those found in Leiden MS 930 containing an Arabic translation of the *Sphaerica* made about A.D. 1007 by al-Bīrūnī's teacher Abū Naṣr Manṣūr (see Suter, 1900, pp. 81 and 225). This was pointed out by Bjørnbo (1902, p. 88). Abū Naṣr Manṣūr's contributions to spherical trigonometry and astronomy have been examined by C. Jensen (1971).

Z. The centre of the sphere is called H. The points A, Δ and Γ define a plane Π_1 containing the chords AΔ, ΔΓ, and ΓA.

Another plane Π_2 is defined by the three coplanar radii HB, HZ, and HE.

Similarly a third plane Π_3 is defined by the radii HA, HΔ, and HB.

The planes Π_1 and Π_2 intersect in the straight line 1. If the chord AΔ and the radius HB are not parallel, they will meet in the point Θ of 1 where Π_1, Π_2, and Π_3 meet. Similarly the chord ΔΓ and the radius HZ will meet in another point K on 1.

Finally AΓ and HE meet in the point Λ of 1.

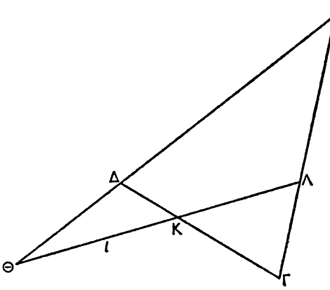


Fig. 3.16

In this way the following plane figure 3.16 is produced. Here Menelaus' theorem in the form (3.32) leads to the relation

$$\frac{\Gamma\Lambda}{\Lambda A} = \frac{\Gamma K}{K\Delta} \cdot \frac{\Delta\Theta}{\Theta A} \tag{3.35}$$

None of the six quantities are chords of a great circle, so that we cannot reduce (3.35) by means of (3.7). Nevertheless, the three ratios can be expressed as ratios between sines by means of the lemma (3.33) or (3.34). This leads to the relation

$$\frac{\sin \widehat{\Gamma E}}{\sin \widehat{EA}} = \frac{\sin \widehat{\Gamma Z}}{\sin \widehat{Z\Delta}} \cdot \frac{\sin \widehat{\Delta B}}{\sin \widehat{BA}} \tag{3.36}$$

which in what follows we shall regard as the spherical equivalent to (3.35). In Ptolemy the ratios of sines are, of course, expressed as ratios of chords, so that (3.36) is found in the form

$$\frac{\mathrm{ch}(2 \cdot \widehat{\Gamma E})}{\mathrm{ch}(2 \cdot \widehat{EA})} = \frac{\mathrm{ch}(2 \cdot \widehat{\Gamma Z})}{\mathrm{ch}(2 \cdot \widehat{Z\Delta})} \cdot \frac{\mathrm{ch}(2 \cdot \widehat{\Delta B})}{\mathrm{ch}(2 \cdot \widehat{BA})} \tag{3.37}$$

which follows from (3.36) and (3.19).

There will be three other possible relations analogous to (3.36) or (3.37). Of these, Ptolemy only states the following (without proof)

$$\frac{\mathrm{ch}(2 \cdot \widehat{\Gamma A})}{\mathrm{ch}(2 \cdot \widehat{AE})} = \frac{\mathrm{ch}(2 \cdot \widehat{\Gamma \Delta})}{\mathrm{ch}(2 \cdot \widehat{\Delta Z})} \cdot \frac{\mathrm{ch}(2 \cdot \widehat{ZB})}{\mathrm{ch}(2 \cdot \widehat{BE})} \tag{3.38}$$

which is equivalent to

$$\frac{\sin \widehat{\Gamma A}}{\sin \widehat{AE}} = \frac{\sin \widehat{\Gamma \Delta}}{\sin \widehat{\Delta Z}} \cdot \frac{\sin \widehat{ZB}}{\sin \widehat{BE}} \tag{3.39}$$

More symmetrical and easily remembered forms of (3.36) and (3.39) are

$$\frac{\sin \widehat{\Gamma Z}}{\sin \widehat{Z\Delta}} \cdot \frac{\sin \widehat{\Delta B}}{\sin \widehat{BA}} \cdot \frac{\sin \widehat{AE}}{\sin \widehat{E\Gamma}} = 1 \tag{3.36 a}$$

and

$$\frac{\sin \widehat{\Gamma \Delta}}{\sin \widehat{\Delta Z}} \cdot \frac{\sin \widehat{ZB}}{\sin \widehat{BE}} \cdot \frac{\sin \widehat{EA}}{\sin \widehat{A\Gamma}} = 1 \tag{3.39 a}$$

Containing no less than six variable quantities, Menelaus' theorem became known to the Middle Ages as *Regula sex quantitatum*. Menelaus had shown how several other theorems of a general nature could be derived from it; but in the *Almagest* Ptolemy always goes back to either (3.37) or (3.38) every time he has to deal with a spherical problem, even if it is a very simple one. Thus he has to remember only one or two fundamental rules; but this also makes his spherical trigonometry less practical – here Zeuthen (1900) is of another opinion – than if he had taken the trouble to apply the theorem to particular classes of simple spherical triangles. The result is that we are in very much the same situation here as in Ptolemy's plane trigonometry (see above p. 65) in so far as we now must try to extract general formulae from the procedures used by Ptolemy for a number of concrete problems in his spherical astronomy the astronomical contents of which will be examined in the following chapter.

Ptolemy considers right-angled spherical triangles only. We know to-day six fundamental relations for such triangles, viz.

$$\sin a = \sin c \cdot \sin A \tag{3.40}$$
$$\tan a = \sin b \cdot \tan A \tag{3.41}$$
$$\cos c = \cos a \cdot \cos b \tag{3.42}$$
$$\tan b = \tan c \cdot \cos A \tag{3.43}$$
$$\cos c = \cot A \cdot \cot B \tag{3.44}$$

and

$$\cos A = \cos a \cdot \sin B \tag{3.45}$$

where as usual the right angle is denoted by C. Of course no such formulae were known to Ptolemy; but the problem is how many of these six relations are implied in the procedures he used in the Almagest (cf. Braunmühl 1900, i, p. 25).

These procedures follow the same general pattern: the given right-angled triangle is made part of a figure (like 3.15) to which the theorem of Menelaus can be applied, either in the form (3.36) or (3.39). Figure 3.17 shows the two ways in which this is done in the Almagest. Here the given triangle is placed either in position 1 or position 2. D and S are chosen in a suitable way (and G is determined accordingly) so that

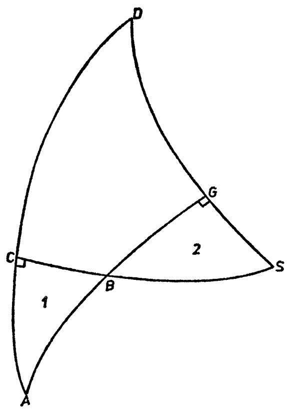


Fig. 3.17

three of the variable quantities in Menelaus' relation disappear. – In the following we shall denote a great circle containing an arc a by C_a.

Case 1. Let us first consider position 1 in connection with (3.36). Here C_a and C_b are perpendicular to each other so that C_b has its poles S (and N) on C_a, and C_a its poles D (and E) on C_b. The poles D and S define a great circle intersected by C_c in G [cf. I, 14; Hei 1, 76]. By (3.36) we get

$$\frac{\sin \overset{\frown}{SD}}{\sin \overset{\frown}{DG}} = \frac{\sin \overset{\frown}{SC}}{\sin \overset{\frown}{CB}} \cdot \frac{\sin \overset{\frown}{BA}}{\sin \overset{\frown}{AG}}$$

The six variables of this relation are reduced to three because we have chosen S and D so that

$$\overset{\frown}{SD} = \overset{\frown}{SC} = \overset{\frown}{AG} = 90°$$

Furthermore we have $\overset{\frown}{DG} = A, \ \overset{\frown}{CB} = a, \ \overset{\frown}{BA} = c$

Substituting these values, we have

$$\frac{1}{\sin A} = \frac{1}{\sin a} \cdot \frac{\sin c}{1}$$

or

$$\sin a = \sin c \cdot \sin A \tag{3.40}$$

and therefore this procedure is equivalent to (3.40).

Case 2. From the same position of the triangle in Figure 3.17 we can find another

of the fundamental relations if we use Menelaus' theorem in the form (3.39) [I, 16; Hei 1, 82]. This gives the relation

$$\frac{\sin \overset{\frown}{SG}}{\sin \overset{\frown}{GD}} = \frac{\sin \overset{\frown}{SB}}{\sin \overset{\frown}{BC}} \cdot \frac{\sin \overset{\frown}{CA}}{\sin \overset{\frown}{AD}}$$

or

$$\frac{\sin (90° - A)}{\sin A} = \frac{\sin (90° - a)}{\sin a} \cdot \frac{\sin b}{1}$$

which is reduced to

$$\tan a = \sin b \cdot \tan A \tag{3.41}$$

Case 3. We now place the given triangle ABC in position 2 in Figure 3.17 and apply (3.39). This gives the relation

$$\frac{\sin \overset{\frown}{BC}}{\sin \overset{\frown}{CS}} = \frac{\sin \overset{\frown}{BA}}{\sin \overset{\frown}{AG}} \cdot \frac{\sin \overset{\frown}{GD}}{\sin \overset{\frown}{DS}}$$

where again

$$\overset{\frown}{CS} = \overset{\frown}{AG} = \overset{\frown}{DS} = 90°$$

This gives

$$\sin \overset{\frown}{BC} = \sin \overset{\frown}{BA} \cdot \sin \overset{\frown}{GD}$$

Remembering that the triangle now given is BGS (with G = 90°) we can make our notation uniform by putting

$$\overset{\frown}{BS} = c; \quad \overset{\frown}{BG} = a; \quad \overset{\frown}{GS} = b$$

We have accordingly

$$\overset{\frown}{BC} = 90° - c$$
$$\overset{\frown}{BA} = 90° - a$$
$$\overset{\frown}{GD} = 90° - b$$

The result is accordingly a relation equivalent to (3.42), viz.

$$\cos c = \cos a \cdot \cos b \tag{3.42}$$

Case 4. Finally, we can derive a fourth relation, using Menelaus' theorem in the form (3.36), on the triangle in position 2. This gives

$$\frac{\sin \overset{\frown}{SG}}{\sin \overset{\frown}{GD}} = \frac{\sin \overset{\frown}{SB}}{\sin \overset{\frown}{BC}} \cdot \frac{\sin \overset{\frown}{CA}}{\sin \overset{\frown}{AD}}$$

where $\widehat{CA} = 90° - A$ and the other arcs are given above. In this way we are led to the relation

$$\frac{\sin b}{\sin(90° - b)} = \frac{\sin c}{\sin(90° - c)} \cdot \frac{\sin(90° - A)}{1}$$

or

$$\tan b = \tan c \cdot \cos A \tag{3.43}$$

Thus four of the six fundamental relations are implied in the Almagest (cf. Braunmühl 1900, p. 25). For oblique-angled spherical triangles Ptolemy proceeds as in plane trigonometry, decomposing them into right-angled parts; examples of this method will be found in the following chapter.

The Mathematics Implicit in the Almagest

In the preface to Book III [Hei 1, 190] of the Almagest Ptolemy makes the explicit assertion that in the preceding two Books he has dealt with all the mathematical prerequisites of the whole study of astronomy and geography. This is a very curious statement, the truth of which we shall examine and, indeed, challenge in the remaining part of this chapter. Its validity depends, of course, very much upon what the word 'mathematical' means in this connection.

In one sense there is no reason to deny Ptolemy's words, which he obviously believed himself. If mathematics is taken as synonymous with geometry then Ptolemy is right, at least to a very large extent, in so far as the Almagest makes use of many of the ordinary parts of traditional Greek geometry. Thus a knowledge of the *Elements* of Euclid is presupposed; and where Ptolemy goes beyond the Elements he carefully explains his methods and proves his own theorems, as we have seen in his development of plane and spherical trigonometry. This reveals that Ptolemy the mathematician conceived of himself as a geometer, and of the Almagest as a work founded upon geometry. This is no wonder, and in complete agreement with the general trend of Greek mathematics from the much earlier times, when the discovery of incommensurable magnitudes drew the attention of mathematicians away from numbers and towards geometry as a field possessed of a more general kind of magnitude[10]). The attitude is in harmony also with what is still postulated in most modern histories of Greek mathematics, with their almost exclusive concentration on the development of geometry.

If, however, we take a broader look at mathematics, then Ptolemy's statement proves as false as the standard historical conception of Greek mathematics. In fact, even a superficial perusal of the Almagest is sufficient to reveal the important fact that geometry was only one part of the theoretical equipment of its author. First, it abounds in numerical calculations made according to methods and procedures which

10) This is the accepted opinion of historians like Zeuthen, Cantor, v. d. Waerden, and others. The view that the discovery of incommensurables provoked a crisis in the history of Greek mathematics has recently been challenged by H.-J. Waschkies, *Eine neue Hypothese zur Entdeckung der inkommensurablen Grössen durch die Griechen*, Truesdell's Archive for History of Exact Sciences, 7, (1970–71), pp. 325–353.

are left unexplained; not only are they not described in the book itself, but as far as we know no Greek mathematician of any repute has ever bothered to give a consistent exposition of such matters. The few examples of practical computations described above were drawn from a much later source (page 52). Second, the very fact that such calculations had to be performed on a large scale, particularly by theoretical astronomers, entailed the consequence that besides purely arithmetical operations we can discern in the Almagest at least the rudiments of a theory of functions. This claim certainly calls for some support before it can be accepted as true; but before we try to confirm it in the rest of this chapter we may reflect a little upon the two following considerations.

First, it seems almost inevitable that by its very nature Greek astronomy would stimulate the development of practical methods for dealing with what we call functions of one or more variables. From very early times Greek astronomers tried to describe the phenomena of the heavens by geometrical models so constructed that they were able to simulate the movements of the planets. To a certain extent the properties of such models could be investigated by purely geometrical methods. But this was only a first step, and at least from the time of Hipparchus (and presumably long before) Greek astronomers set themselves a much greater task. Not satisfied with knowing the abstract behaviour of the models, they tried to provide them with numerical parameters in such a way that the position of the Sun, Moon and Planets could be predicted for any given time (page 267). Such a purpose could not be without mathematical consequences.

The configuration of a geometrical model for the motion of a planet changes continuously with time. At any definite point of time it can be determined by ordinary geometrical rules leading to procedures for calculating the corresponding apparent position of the planet as expressed by a pair of celestial coordinates, such as, for instance, its longitude and latitude. When this has to be possible at any given time it clearly implies the use of standard and general algebraic methods which can be applied regardless of the particular time for which the calculation is carried out[11]). Such standard procedures are, in fact, very common in the Almagest, where they are explained in one of the two following ways.

Usually a procedure of this kind is illustrated by a numerical example showing how an actual calculation is carried out, the general character of the method employed being either postulated or tacitly assumed. But in other cases Ptolemy gives a verbal description of the procedure, using general terms but no particular numerical values. This results in a number of what may be called 'programmes' similar in many respects to the programmes fed into a modern computer. There are

11) Another but less accurate way of dealing with a continuously variable geometrical model would be to construct a mechanical instrument simulating the behaviour of the model. This led to various types of *planetaria* of which some at least were invented in Antiquity by Archimedes (*Works*, transl. Heath, p. xxi), Heron (Drachmann 1971), and others. It seems that such instruments were not used for computations, unlike the Mediaeval *aequatorea* which were real analogue computers; their prototype is the device described by Proclus (*Hyp.* III, 4, p. 73), cf. note 12.1, page 355.

many instances of such programmes in the Almagest; they are usually found in particular chapters at the end of the various books, where Ptolemy summarizes the results of his investigation of a certain model and shows how it is made fit for actual calculations. To mention but a few examples, there is a verbal programme of how to calculate the longitude of the Sun [III, 8; Hei 1, 257, cf. below page 153] and the Moon [V, 9; Hei 1, 392, cf. page 195] at a given time, and also the parallax of the Moon [V, 19; Hei 1, 444, cf. page 214].

It is obvious that the use of a standard procedure of this kind presupposes the existence of a functional relationship between two sets of variables. Here one must keep in mind that the word 'function' was unknown to ancient mathematics. It belongs, indeed, to a much later age. But are we for that reason justified in concluding that they had no idea of functional relationships? The answer to that question depends very much on how these latter terms are understood.

If we, with many earlier mathematicians[12]), conceive a function as essentially a formula combining variables by means of a conventional notation, then it is true to say that the mathematicians of Antiquity had no functions, for the very simple reason that they were not possessed of a mathematical notation enabling them to write such formalized expressions. This applies also to Ptolemy, whose mathematical notation is neither more nor less developed than that of Euclid. Thus letters stand for points, and combinations of letters for lines and angles, but never for numbers. This explains why the standard computational procedures employed in the Almagest are given the form of verbal statements, but not of formal expressions in a symbolic language. It will appear later that many of the procedures derived from the Ptolemaic models of planetary motion would, in fact, lead to extremely complicated expressions if they were transposed into modern mathematical notation.

But if we conceive a function not as a formula, but as a more general relation associating the elements of one set of numbers (for example points of time $t_1, t_2, t_3 \dots$) with the elements of another set (for example planetary longitudes $\lambda_1, \lambda_2, \lambda_3 \dots$) then it is obvious that functions in this sense abound throughout the Almagest[13]).

12) Here one might quote Euler (1748, I, 4) who gives the following definition: *Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili et numeris seu quantitatibus constantibus*, continuing with a classification of functions based on their analytical expressions. This concept went into many text-books, and was popularized by *L'Encyclopédie* (vol. XIV, Lausanne et Berne 1779, p. 855). Montucla (Histoire des Mathématiques III, nouv. éd. Paris, An X (1802), p. 265) gave an even more restrictive definition of a function as an *expression algébrique*.

13) It is customary to mention Dirichlet (1837) as the originator of this concept of function: *Man denke sich unter a und b zwei feste Werthe und unter x eine veränderliche Grösse, welche nach und nach alle zwischen a und b liegenden Werthe annehmen soll. Entspricht nun jedem x ein einziges, endliches y, und zwar so, dass, während x das Intervall von a bis b stetig durchläuft, $y = f(x)$ sich ebenfalls allmählich verändert, so heisst y eine stetige oder continuirliche Function von x für dieses Intervall. Es ist dabei gar nicht nöthig, dass y in diesem ganzen Intervalle nach demselben Gesetze von x abhängig sei, ja man braucht nicht einmal an eine durch mathematische Operationen ausdrückbare Abhängigkeit zu denken.* It is worth noticing, however, that already S. F. Lacroix (1797) came very near to the new idea: *Toute quantité dont la valeur dépend d'une ou de plusieurs autres quantités, est dite fonction de ces dernières, soit qu'un sache ou qu'on ignore par quelles opérations il faut passer pour remonter de celles-ci à la première.*

Only the word is missing – the thing itself is there and clearly represented by the many tables of corresponding elements of such sets. To substantiate this assertion we shall now consider first some of the more typical functional relationships used by Ptolemy, and next see how far and in which manner he was able to deal with problems involving questions of continuity and differentiability.

Functions of One Variable

Functions of one variable play a particular role in the Almagest, first because they occupy an important position in all the theories of longitude, and second because they serve as a means of handling functions of two or more variables by various methods described below (page 84). Here we must distinguish between algebraic and trigonometrical functions.

The only purely *algebraic functions* found in the Almagest are *linear* relationships. As we are going to see (page 291), the longitude of any of the planets is found by adding certain corrections (called 'equations') to a mean longitude which according to (5.2), (6.2) or (9.12) can be written in the form

$$\lambda_m(t) = \lambda_m(t_0) + \omega(t - t_0) \tag{3.46}$$

Here $\lambda_m(t_0)$ is a constant longitude, ω a constant mean angular velocity, t the time, and t_0 a fixed epoch. This indicates that a linear function of the type

$$y = ax + b \tag{3.46 a}$$

is fundamental to Ptolemaic astronomy. But for some reason or other Ptolemy consistently avoids tabulating (3.46) in the form of an ephemeris of the actual mean longitude. Instead he tabulates the variable term $\omega(t - t_0)$ as a function of $(t - t_0)$. Accordingly the only linear function tabulated in the Almagest is of the simplest possible form

$$y = ax \tag{3.46 b}$$

Among the many *trigonometrical functions* of one variable found in the Almagest we have already studied the function of chords

$$\text{ch}(x) = 2R \sin \frac{x}{2} \quad \text{when } 0° \leqslant x \leqslant 180° \tag{3.47}$$

giving the length of a chord of an arc of x degrees in a circle of radius R (see page 63). This seems to be the first trigonometrical function tabulated in the history of Greek mathematics, and it is often said that it is the only one represented in the Almagest. But this is a too restrictive point of view.

Let us first consider the fact that the relationship (3.47) is expressed in the form of a table with two columns according to the following scheme (a third column containing differences need not concern us here). In Ptolemy's own account of this table

I	II
x_1	$ch(x_1)$
x_2	$ch(x_2)$
x_3	$ch(x_3)$
.	.
.	.
.	.

[I, 10; Hei 1, 47] the arcs in Column I are considered as representing what we call the independent variable while the chords in Column II are the corresponding values of the dependent variable. Passing from Col. I to Col. II is therefore equivalent to determining a chord from its arc. Now it is important to notice that in his practical calculations Ptolemy quite as often passes from Col. II to Col. I, that is, he uses the table to find the arc from its chord. Accordingly we must conclude that if we regard the Table of Chords as expressing the function (3.47) then we must necessarily regard it as representing also the inverse function

$$x = 2 \arcsin \frac{ch(x)}{2R} \qquad (3.48)$$

Ptolemy never comments upon these two ways of using the table. In fact, they are different only if we regard (3.47) and (3.48) as different functions expressed by two different formulae. But if we regard them as different formalized expressions of the same functional relationship between the two sets of values listed in the table then there is no particular reason for distinguishing between a function and its inverse function. In other words, the fact that all Ptolemy's functions are represented by tables instead of formulae has the consequence that the problem of inverting a function of one variable does not arise.

The literature on the history of trigonometry usually underlines that the Table of Chords is the only purely trigonometrical table in the Almagest, with the additional remark that Ptolemy had no knowledge of any of the fundamental trigonometrical functions. The first part of this statement is true while the second part of it needs some modification. Thus the table of the various equations of the Moon (see page 197) contains a final column VII in which the latitude $\beta_☽$ of the Moon is tabulated as a function of an independent variable λ_d according to the relation

$$\beta_☽ = 5° \cdot \sin \lambda_d \qquad (6.65)$$

Considered in itself such a table does not entitle us to say more than that there exists a certain latitude function (6.65) in the lunar theory. But it is interesting to note that in the theory of the latitudes of the superior planets (page 368) Ptolemy introduces a factor defined by

$$0;12 \cdot \beta_☽ = \frac{\beta_☽}{5} = \sin \lambda_d \qquad (12.28)$$

The same factor appears also in the theory of the latitudes of Venus and Mercury and is tabulated in Column V of the general table of latitudes [XIII, 5; Hei 2, 582]. Accordingly we must conclude that even if Ptolemy did not have the sine function in the sense in which it is defined by its abstract trigonometrical properties[14]), nevertheless the Almagest contains a real *table of sines* expressed as sexagesimal fractions (minutes and seconds of unity).

Two other functions of one variable are found in the spherical astronomy of the Almagest described in the following chapter. The first is based on the relation

$$\sin \delta(\lambda) = \sin \varepsilon \cdot \sin \lambda \tag{4.2}$$

Here $\delta(\lambda)$ is the declination of a point on the ecliptic having the longitude λ from the vernal equinox, and ε is the constant angle between the ecliptic and the equator. The so-called *Table of the Obliquity of the Ecliptic* [I, 15; Hei 1, 80] gives δ as a function of λ and vice versa.

The second of these functions is determined by the relation

$$\sin \alpha(\lambda) = \cot \varepsilon \cdot \tan \delta(\lambda) \tag{4.3}$$

where $\alpha(\lambda)$ is the right ascension of a point on the ecliptic with the longitude λ and the declination $\delta(\lambda)$ given by (4.2). This relationship is treated in a quite different way since the table [I, 16; Hei 1, 85] does not simply list $\alpha(\lambda)$ as a function of λ, but the successive differences of both variables. It is constructed by dividing the interval $0° \leqslant \lambda \leqslant 90°$ into 9 equal parts by the points λ_n, $n = 0, 1 \ldots 9$ and tabulating $(\lambda_n - \lambda_{n-1})$ against $(\alpha(\lambda_n) - \alpha(\lambda_{n-1}))$. We notice that Ptolemy here makes use of the symmetrical properties of (4.3); this dispenses him with considering the function throughout the complete interval $0° \leqslant \lambda \leqslant 360°$. Such use of symmetry is made whenever it is possible.

Our two final examples of trigonometrical functions are a little more sophisticated. Thus in the theory of parallax we find a relation (7.12) of the type

$$\sin y = \frac{\sin x}{a - \cos x} \tag{3.49}$$

where a is a positive constant > 1. It is tabulated with y as a function of x in the first two columns of the general table of parallaxes [V, 18; Hei 1, 442], to which we shall revert in more detail at a later stage (page 217). The values of y are determined as

$$y = \arcsin \left(\frac{\sin x}{a - \cos x}\right)$$

by means of the table of chords. This simple procedure is not possible with the relation

$$\tan y = \frac{a \sin x}{b + a \cos x} \tag{3.50}$$

14) In this sense the function seems to have been introduced in India where it appears already in the Surya Siddhānta; see Kennedy (1969) p. 346.

(where a and $b > a$ are positive constants) which appears in the form (5.28) as the so-called equation of the Sun, and also as the equation of the Moon (6.29) in the first model of the Moon's motion. To construct corresponding values of x and y Ptolemy is forced to replace (3.50) by the equivalent formula

$$\sin y = \frac{a \sin x}{\sqrt{(b + a \cos x)^2 + (a \sin x)^2}} \tag{3.51}$$

which gives y as a function of x by means of the table of chords. An even more complicated relation of the same type is found in the theory of the superior planets as (9.23), which is of the form

$$\sin y = \frac{2a \sin x}{\rho(x)} \tag{3.52}$$

where

$$\rho(x) = [\{\sqrt{b^2 - (a \sin x)^2} + a \cos x\}^2 + \{2a \sin x\}^2]^{\frac{1}{2}} \tag{3.53}$$

Here y is tabulated directly as a function of x, while $\rho(x)$ does not appear in tabular form.

Linear Interpolation

All the functions of one variable mentioned in the examples above happen to be continuous. It goes without saying that the mathematical notion of continuity as an explicit concept is unknown to Ptolemy. That he, in fact, treats these functions as continuous appears from his unspoken presupposition that it is possible to determine a value of the dependent variable corresponding to any value of the independent variable by the simple process of linear interpolation according to a procedure corresponding to the formula

$$f(x) = f(x_n) + [f(x_{n+1}) - f(x_n)] \cdot \frac{x - x_n}{x_{n+1} - x_n} \tag{3.54}$$

where x is an arbitrary value of the independent variable lying in the interval

$$x_{n+1} > x > x_n$$

between two consecutive tabulated values x_n and x_{n+1}.

Functions of Two Variables

The first tabulated function of two variables found in the Almagest is described in the book on spherical astronomy [II, 7; Hei 1, 117]. The problem is to find the rising time of an arc on the ecliptic, or to find the length of that arc on the equator which rises above the horizon together with a given arc on the ecliptic. As shown below

(page 110), Ptolemy's solution of this problem is equivalent to the formula (4.22) in which $\vartheta(\lambda, \varphi)$ is a function of the two variables λ and φ.

A function of this kind can be represented by a table with double entry constructed according to the following general scheme

	φ_1	φ_2	φ_3	φ_4 ....
λ_1	$\vartheta(\lambda_1, \varphi_1)$	$\vartheta(\lambda_1, \varphi_2)$	$\vartheta(\lambda_1, \varphi_3)$	....
λ_2	$\vartheta(\lambda_2, \varphi_1)$	$\vartheta(\lambda_2, \varphi_2)$	$\vartheta(\lambda_2, \varphi_3)$	....
λ_3	$\vartheta(\lambda_3, \varphi_1)$	$\vartheta(\lambda_3, \varphi_2)$	$\vartheta(\lambda_3, \varphi_3)$	....
λ_4	.	.	.	
.	.	.	.	
.	.	.	.	
.	.	.	.	
.	.	.	.	

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ is a discrete set of values of the variable λ, and $\varphi_1, \varphi_2, \varphi_3, \ldots$ is another discrete set of values of the variable φ. This is the fundamental pattern of Ptolemy's so-called *Table of Oblique Ascensions* [II, 8; Hei 1, 134], which is calculated for 36 values of λ, and for 11 specially selected values of φ (see page 111). This gives a total of 396 tabulated values of the function $\vartheta(\lambda, \varphi)$ and thus a very large table; the fact that it is further enlarged by the introduction of an extra column of differences $\{\vartheta(\lambda_n, \varphi_i) - \vartheta(\lambda_{n-1}, \varphi_i)\}$ for each value of φ_i is without importance for the present investigation.

Values of $\vartheta(\lambda, \varphi)$ not listed in the table are here determined by linear interpolation with respect to both the independent variables.

It goes without saying that the construction of a table with a double entry representing a function of two variables involves a great amount of calculation. One of the most interesting features of Ptolemy's 'theoıy of functions' is the fact that he realized how this work could be greatly reduced if the function had certain special properties. The result is a particular method of interpolation which we shall analyse in the following section.

Ptolemy's Method of Interpolation

To understand this method we may begin by distinguishing between a strong and a weak independent variable. Let $f(x, y)$ be a function of the variables x and y defined in the closed intervals

$$a \leqslant x \leqslant b \quad \text{and} \quad c \leqslant y \leqslant d$$

If we have

$$|f(x, y_1) - f(x, y_2)| \quad \leqslant \quad |f(x, c)| \tag{3.55}$$

for all values of x in the interval $a \leqslant x \leqslant b$ and for any two values y_1 and y_2 of y in the interval $c \leqslant y \leqslant d$ we shall call y a weak variable. If the relation (3.56) is not satisfied we shall call y a strong variable. Similar definitions apply, of course, to the other variable x. This distinction can now be applied to the numerous functions of two variables found in the Almagest. It then appears that, for instance, in the function (4.22) both the variables λ and φ are strong; in such a case Ptolemy cannot avoid using a table with two separate entries to represent the function. But if there is one strong and one weak variable the method of representation can be greatly simplified.

Let us again consider a function $f(x, y)$ defined in the intervals

$$a \leqslant x \leqslant b \quad \text{and} \quad c \leqslant y \leqslant d$$

and let us assume that

1) x is a strong and y is a weak variable in the sense defined above, and
2) $f(x, y)$ is a monotone increasing function of y, so that we have for all values of x

$$f(x, d) - f(x, c) > 0 \tag{3.56}$$

Ptolemy's method of interpolation can then be summarized by the formula

$$f(x, y) = f(x, c) + [f(x, d) - f(x, c)] \cdot g(y) \tag{3.57}$$

valid for all values of x and y inside the given intervals. Here $g(y)$ is an interpolation function which remains to be determined. It follows from Ptolemy's numerical procedures that he puts

$$g(y) = \frac{\max_x f(x, y) - \max_x f(x, c)}{\max_x f(x, d) - \max_x f(x, c)} \tag{3.58}$$

where

$$\max_x f(x, y) \tag{3.59}$$

means the maximum value of $f(x, y)$ considered as a function of x only for a fixed value of y.

The formulae (3.57) and (3.58) characterize a specific method which we shall call *Ptolemaic interpolation*. It is a standard procedure in the Almagest and allows us to reduce a function of two variables satisfying the conditions 1) and 2) above to an algebraic expression in functions of one variable, each of which can be tabulated in the usual way.

A minor characteristic of Ptolemy's many tables of functions of two variables may be noticed here. In the numerous cases where x and y are angular variables the symmetrical properties of the functions will often entail that the intervals $(a \mid b)$ and $(c \mid d)$ are identical. In this case the tabulated set of x-values can be chosen as identical to the tabulated set of y-values, so that the table can be provided with a column of common arguments. A typical arrangement is the following system of 4 columns

Column I: common arguments
Column II: $f(x, c)$
Column III: $[f(x, d) - f(x, c)]$
Column IV: $g(y)$

Here the arguments in Column I are taken as values of x in connection with Columns II and III, but as values of y in connection with Column IV. This scheme is used, for instance, in most of the tables of equations.

It is proper once more to underline that this method of dealing with functions of 2 variables is presented everywhere in the Almagest by a number of purely numerical examples of which (3.57) is a formalized summary, and also that the important interpolation function (3.58) is introduced in the same way without any kind of proof. It is not difficult to give a plausible reconstruction of the reasoning which might have led Ptolemy to (3.57). Here the first term $f(x, c)$ represents $f(x, y)$ for an arbitrary value of the strong variable, and at the lower end of the interval $(c \mid d)$ inside which the weak variable y is allowed to vary. The total increase of $f(x, y)$ when y goes from c to d is $[f(x, d) - f(x, c)]$. If y goes from c to an intermediate value then the increase will be a certain fraction of $[f(x, d) - f(x, c)]$, and this fraction must be a function $g(y)$ of the weak variable y only. This explains the general structure of (3.57). Furthermore it entails the consequence that

$$g(c) = 0$$
$$g(d) = 1$$

in agreement with (3.58).

One might have expected Ptolemy to use linear interpolation also in this case, i.e. to define the interpolation function in (3.57) by

$$g(y) = \frac{y - c}{d - c} \tag{3.60}$$

instead of by the more complicated expression (3.58). Why this expression was chosen is not explained in the Almagest, and Ptolemy does not give even the barest indication of how he arrived at it. A tentative reconstruction of his line of reasoning can accordingly be nothing more than a mere conjecture; but it is only natural to suspect that (3.58) was derived in connection with that particular kind of problem to which it was most often applied. This problem appears in the epicyclic models of the planetary theories. These will be explained in the following chapters 6,9, and 10, where we often find a function defined by a relation like (9.40) or similar expressions, which we can write in the general form

$$f(x, y) = \arcsin\left(\frac{r \sin x}{\Delta(x, y)}\right) \tag{3.61}$$

where $\Delta(x, y)$ is a slowly varying function of both x and y. In geometrical terms $f(x, y)$ can be interpreted as the so-called *prosthaphairesis* angle, or equation of

argument, that is the angle CTP under which an observer at the centre T of the Earth would see the radius CP from the centre C of the epicycle to the actual position P of the planet. The complete model is shown in Figure 9.7, where an epicycle of constant radius CP = r rides upon an eccentric deferent circle with centre D outside T.

One consequence of this model is that if the epicycle itself were a visible circle then it would seem (to an observer at T) to increase in size when its centre is drawn nearer to the Earth. Consequently this model could be replaced by another, having a concentric deferent and an epicycle of variable size as shown in Figure 3.18.

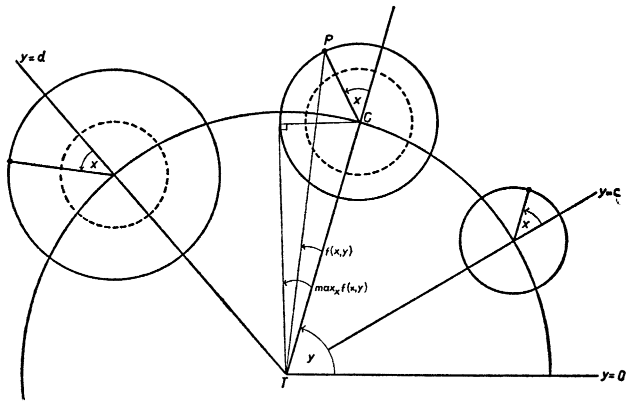


Fig. 3.18

Here we have supposed $\Delta(x, y)$ in (3.61) to be a slowly decreasing, monotone function of y for x constant in the interval from y = c to y = d, y being an angular coordinate defining the position of C on the deferent while x defines the position of P on the epicycle. This supposition entails that the size of the epicycle will increase as y goes from y = c to y = d. Since f(x, y) given by (3.61) measures the angle CTP, it follows that the instantaneous radius of the epicycle is $\max_x f(x, y)$ for any intermediate value of y.

It also follows that the interpolation function (3.58) can be interpreted as the relative increase of the apparent size of the epicycle. Therefore it is clear that the assumption underlying the interpolation formula (3.57) can be stated as follows:

> For any fixed position (x = constant) of the planet upon the epicycle the angle CTP = f(x, y) varies in the same proportion as the apparent size of the epicycle itself.

This is, of course, no proof of the Ptolemaic method of interpolation, but only a restatement of (3.57) in terms of the geometrical models of the motions of the planets. It is in agreement with a particular representation of these models used by Latin astronomers of the Middle Ages, who often denoted the difference [f(x, d) − f(x, c)] in (3.57) by *diversitas diametri circuli brevis*; this reveals that they had an epicycle

(*circulus brevis*) of variable size in mind[15]). That Ptolemy himself derived (3.58) by the same device is at least a plausible hypothesis.

Functions of Three Variables

The methods described above can be extended to functions of 3 variables, of which there are two examples in the Almagest. In the following chapter (page 118) we shall see how Ptolemy determines the angle between the ecliptic and a vertical circle as a function $\psi(\varphi, t, \lambda)$ of the geographical latitude φ, the hour t, and the longitude λ of the intersection between the two circles [II, 12; Hei 1, 160]. Since all the three independent variables are strong, Ptolemy deals with this function by means of a table with three entrances [II, 13; Hei 1, 174].

The second example is found in the general theory of the parallax of the Moon [V, 17; Hei 1, 428]. As we are going to see (page 214) this leads to a parallax function $\pi(z', a_v, c)$ (7.13) of 3 variables – the theoretical (geocentric) zenith distance z' of the Moon, its true anomaly a_v defining its position on the epicycle and given by (6.43), and the centrum c given by (6.41) and determining the position of the epicycle centre. Of these variables z' is the strongest, and c the weakest. In this case Ptolemy is able to extend the interpolation method described above (page 86) from 2 to 3 variables, tabulating π as a function of z' and introducing corrections dependent on the values of a_v and c in analogy with (3.57). The method will be examined in more detail below (page 214) and since it contains no essentially new features we shall not here consider it further.

Extremal Properties

Expressions like (3.59) raise the question of how Ptolemy is able to determine the maximum value of a given function. Here there is no trace of any general method in the Almagest. The first instance where an extremal problem is solved occurs in the theory of the motion of the Sun [III, 3; Hei 1, 222], where Ptolemy proves that a particular function – the equation of centre q(a) given by (5.26) – of the general type (3.50) is maximal at a certain position of the Sun. The proof will be analysed later (page 141) and here we notice only that it rests upon the two implications

$$\begin{aligned} x < x_0 \quad &\Rightarrow \quad f(x) < f(x_0) \\ x > x_0 \quad &\Rightarrow \quad f(x) < f(x_0) \end{aligned} \tag{3.62}$$

both of which are proved by a beautiful, purely geometrical argument, preceded by several auxiliary lemmas. A similar proof of an extremum – the stationary points of

15) Thus the 13th century, anonymous *Theorica planetarum*: *Et sciendum est, quod maiores sunt equaciones argumenti centro epicicli existente in opposito augis ecentrici quam in auge, et differencia, que est inter has equaciones argumenti centro epicicli existente in auge et in opposito augis, dicitur equacio diuersitatis dyametri circuli brevis.*

a planet – is found in the theory of retrograde motions [XII, 1; Hei 2, 459] and is described in detail in Chapter 11 (page 335). Also the maximum value of (3.61) is determined by a geometrical and trigonometrical method.

Such examples show that Ptolemy was capable of determining extrema of a number of special functions by exact methods. The exactness of the proofs depends, however, upon their synthetic, geometrical character, which entails that they cannot be generalized. Moreover, they presuppose the knowledge of the value $x = x_0$ producing the extremum $\max f(x) = f(x_0)$, but there is no indication of how x_0 can be found. Yet it is unlikely that Ptolemy had to rely completely on intuition. The functions he used in the Almagest are represented by tables, and in many cases a simple inspection of a table will suffice to find an approximate value of $\max f(x)$ and also the corresponding value x_0 of the independent variable.

In the theory of the Sun Ptolemy not only proves that the equation of centre is maximal at a given position. He also maintains that at this same position the true (instantaneous) angular velocity ω of the Sun is equal to its mean angular velocity $\omega_\odot$ given by (5.1) [III, 3; Hei 1, 220]. This assertion is analysed in Chapter 5 (page 143) and here we shall anticipate only the result, viz. that around a maximum the variation of a function is negligible. Ptolemy does not state this important property of differentiable functions in so many words; but it is impossible to escape the conclusion that his familiarity with tabulated functions must have led him to an intuitive awareness of this fact.

The Derivative of a Function

That Ptolemy did not succeed in creating universal methods for determining maxima of functions in general is of course due to the fact that he had no proper mathematical analysis at his disposal. Therefore, it is all the more remarkable that the Almagest contains a general procedure for determining the instantaneous angular velocity of a celestial body. The best example of such a procedure is found in the theory of conjunctions of the Sun and Moon [VI, 4; Hei 1, 474] as part of the theory of eclipses. As we are going to see (page 226), Ptolemy relies here on a simple geometrical model, according to which the longitude of the Moon at any given time t is expressed as

$$\lambda(t) = \lambda_m(t) + p(a) \tag{3.63}$$

Here $\lambda_m(t)$ is the mean motus of the Moon given by (3.46), and $p(a)$ is a prosthaphairesis function of an independent variable a, which again is a linear function of time, of the form

$$a(t) = a(t_0) + \omega_a(t - t_0) \tag{3.64}$$

To find the instantaneous angular velocity we differentiate (3.63) with respect to time:

$$\omega(t) = \frac{d\lambda(t)}{dt} = \frac{d\lambda_m}{dt} + \frac{dp}{da} \cdot \frac{da}{dt}$$

which reduces to

$$\omega(t) = \omega + \frac{dp}{da} \cdot \omega_a \tag{3.65}$$

Now the procedure defined by Ptolemy is equivalent to the formula

$$\omega(t) = \omega + \frac{p(a_n) - p(a_n - d)}{d} \cdot \omega_a \tag{3.66}$$

where a_n is one of the tabulated arguments of the function p(a), and d is the difference between two successive arguments in the table of p(a). It appears that (3.65) and (3.66) would be identical if we had

$$\frac{dp}{da} = \frac{p(a_n) - p(a_n - d)}{d} \tag{3.67}$$

This shows that Ptolemy's procedure is sound and would have led to the derived function dp/da if only he had been able to introduce the notion of limits into (3.67). This he was unable to do, and the right hand side of (3.67) is properly described only as *the* [increase of the] *prosthaphairesis corresponding to* [an increase of] *one degree in longitude* [cf. XII, 2; Hei 2, 468]. But although the concept of limits is absent from Ptolemy's procedure, nevertheless it provided him with an approximate method for calculating numerical values of an instantaneous velocity at any given point of time[16]).

In mechanics the concept of an instantaneous velocity remained more or less of an intuitive nature until the discovery of mathematical analysis in the 17th century. It is therefore worth noticing that astronomers from Ptolemy onwards – and perhaps even before – were possessed of practical numerical methods for determining such velocities. This sheds an interesting light not only upon their practical mathematical methods, but also upon the relations between astronomy and mechanics. Had these relations been closer, Ancient and Mediaeval mechanics might have profited from the kinematics developed in astronomy.

Envoi

The preceding brief and sketchy account of the mathematics implicit in the Almagest shows that Ptolemy was wrong in maintaining that the mathematical prerequisites of astronomy were exhausted by the geometry and trigonometry of Book II. On the contrary, Ptolemy presupposes a great number of practical methods for dealing with functions of many different kinds. These methods are indeed absolutely essential to

16) This seems, in general, to have escaped the attention of historians of mechanics. Thus R. Dugas, *Histoire de la mécanique*, Neuchâtel, 1950, has a chapter on *Sources Alexandrines* in which Ptolemy is not mentioned. Neither does his name appear in the long *excursus* on Greek kinematics in M. Clagett, *The Science of Mechanics in the Middle Ages*, Madison, 1959, 163–184, although the author is aware of the role played by astronomy in the development of kinematics.

Ptolemaic astronomy[17]), which without them would have been completely unable to cope with the general problem of determining planetary positions at any given time (see page 267).

The fact that Ptolemy takes this practice of functions for granted points to the conclusion that such methods were not completely unknown in his time, at least among astronomers, and not his own invention. It must be a matter of further research to discover their origin and development before Ptolemy, and in particular to investigate their possible connection with the mathematical methods of Babylonian astronomy. This task falls outside the scope of the present work. But one can surmise that it will not be an easy one because of the lack of earlier sources. As mentioned above (page 78 f.), Greek mathematical literature is completely silent on the theory or practice of functions, at least as far as it has come down to us. But since also Greek historians of mathematics omit the subject, it is unlikely that there ever existed a Greek treatise devoted to this branch of mathematics.

The question is why professional Greek mathematicians chose to ignore a subject of the utmost importance to mathematical astronomy? This is all the more mysterious in view of the fact that they devoted much labour to the equally important subjects of spherical geometry and trigonometry. To point out that Greek mathematics had become more and more synonymous with geometry is not a completely satisfactory answer, since the Greeks did in fact write treatises on arithmetic. A more promising answer is that this cryptic branch of mathematics must have been extremely objectionable to any Greek mathematician educated in the classical tradition of Euclid, Archimedes, and Apollonius. As we have seen, Ptolemy's methods of dealing with functions constituted more a practical art than a theoretical science. It had no proper theorems at all, but only practical rules. Also its fundamental presuppositions, or axioms, remained obscure. In other words, it had not achieved the form and status of a logical-deductive system, in marked contrast to the logically highly advanced science of geometry.

This may well explain why functional mathematics must have seemed an unworthy subject in the state in which it was cultivated by astronomers. But then another question arises. Why did Greek mathematicians never try to improve the quality of this despised subject? Again we have no literary evidence to provide a satisfactory answer. One major obstacle to such an undertaking would be the approximate character of the whole matter, but this is once more an inadequate reason. Strictly speaking, the Greek ideal of mathematics as a system of eternal truth of an unchanging and exact character should not have been able to prevent a serious effort to give this numerical doctrine a properly mathematical status; after all, one had been able to abstract an ideal geometrical system from the most imperfect geometrical properties

17) That the idea of functional relationships was utilized outside the field of astronomy has been demonstrated by Schramm (1965) who examined such relationships in Ptolemy's *Optics*, particularly in connection with the law of optical refraction (Schramm also stressed the importance of astronomical tables, but only in the context of Muslim astronomy). Sambursky (1963 p. 76–78) has drawn attention to a passage in Johannes Philoponos' commentary to Aristotle's *De generatione et corruptione* describing (in qualitative terms) the variation of a function of two variables.

of material things; this was at least Aristotle's theory. Why not try to formulate eternal truths about the properties of functional relationships on the basis of their imperfect and approximate representations in purely numerical tables?

Here again we are at a loss. Leaving all speculative answers aside, we must conclude that hidden away behind the geometrical facade of official Greek mathematics there existed a well established practical art of dealing with many problems of functional relationship, that this area was ignored by mathematicians, but cultivated by astronomers, enabling them to attain the greatest intellectual achievement in Greek science, and that in consequence a study of the mathematical methods of Greek astronomy is essential for a more complete account of the scope and methods of Greek mathematics than that which modern historical expositions usually provide.

CHAPTER 4

Spherical Astronomy in the Almagest

Introduction

In Chapter 3 our exposition was unhistorical in so far as Ptolemy did not present his spherical trigonometry as an abstract mathematical theory. He developed it in close connection with spherical astronomy, and we dealt with the two subjects separately only in order to get a coherent survey of the complete mathematical apparatus implicit in the Almagest. Accordingly we must now consider the astronomical aspect of Ptolemy's spherical trigonometry, examining a series of problems connected with the geometry of the celestial sphere and its diurnal revolution. Only the main problems are investigated while a number of minor corollaries, or theorems of secondary importance are left out.

Before entering upon details a few general remarks are called for. Thus it is a remarkable fact that the Almagest does not define the fundamental concept of spherical astronomy. Therefore, it contains no 'spherica' in the sense in which this notion had become common in Greek astronomical literature (see above, page 73). The subject is not developed from its beginning, nor presented in an axiomatic way. On the contrary, Ptolemy assumes his readers to be acquainted with the definitions of the standard circles and coordinates of the celestial sphere, presumably from the study of such manuals as were mentioned in the previous chapters. However, he does not refer to any particular work by Autolycus, Euclid, or Theodosius, but concentrates upon what can be added to such introductory writings by spherical-trigonometrical methods. Actually, Book II of the Almagest is almost exclusively devoted to showing how known geometrical propositions in spherical astronomy can be subjected to numerical calculation. There is no doubt that some of this work goes back to Hipparchus, and it is difficult to separate Ptolemy's own achievements from those of his great predecessors.

Such calculations usually involve a number of variables some of which are identical with celestial coordinates as used to-day. Here we must remember that the Almagest is primarily an exposition of planetary theory, describing the motions of the planets by means of geometrical models. Since all these motions take place in the neighbourhood of the ecliptic we understand why the ecliptic longitude (in the following denoted by λ) and the ecliptic latitude (denoted by β) are the coordinates most widely used by Ptolemy. Less fundamental are coordinates referring to the celestial equator (declination, right and oblique ascension) or to the horizon.

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_4,

The spherical astronomy of the Almagest is presented in the form of a series of problems the most important of which we have numbered from 1 to 11. They can be classed under four headings:

A: Transformation of coordinates (1–3)
B: Astronomical geography (4–7)
C: Rising and setting of the stars (8)
D: The instantaneous position of the ecliptic (9–11)

In the figures used in the following we have, in most cases, retained the Greek letters used by Ptolemy to denote points and lines. In the Almagest the heavenly sphere and its circles are drawn in many ways, but Ptolemy often draws them as seen from the East.

A: Transformation of Coordinates

Problem 1: To find the declination of a point on the ecliptic.

This is the first of the problems in spherical astronomy dealt with in the Almagest [I, 14; Hei 1, 76]. It presupposes that we know the obliquity ε of the ecliptic, i.e. the angle between the ecliptic and the celestial equator. In a previous chapter [I, 12; Hei 1, 64] Ptolemy has described how this parameter can be measured[1]) in two different ways, with the result that

$$47°;40 < 2\varepsilon < 47°;45$$

from which follows a mean value of

$$\varepsilon \approx 23°;51,15$$

Ptolemy remarks that this agrees very well with the value determined by Eratosthenes, viz.

$$2\varepsilon = 11/83 \text{ of the meridian,}$$

or

$$\varepsilon = 23°;51,20 \tag{4.1}$$

a value which according to Ptolemy was also used by Hipparchus.

1) A description of Ptolemy's instrument – a meridian quadrant or 'plinth' – together with a discussion of his method and result has been given by Britton (1969). At Ptolemy's time the actual value was $\varepsilon = 23°;40,50$. The error could arise from the difficulty of determining noon, or observing the shadow at noon; Britton arrives at no definite conclusion, but refutes the accusation made by Delambre (1817, ii, 75 ff.) that Ptolemy faked or tampered with his observations. – Concerning a slightly later Chinese determination leading to almost the same result, see Hartner (1954).

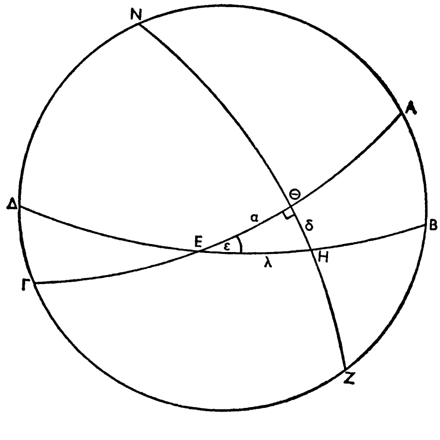


Fig. 4.1

Figure 4.1 shows the equator AEΓ with its poles N and Z, and the ecliptic ΔEB. The great circle ZBAN is one of the colures (i.e. great circles through the poles of the equator and the four main points of the ecliptic, viz. the two equinoctial and the two solstitial points). The point E is the autumnal equinox from which the longitude λ(H) of a point H on the ecliptic is reckoned in this problem; there are other problems in which Ptolemy reckons longitudes as we do to-day, that is from the vernal equinox. In the example chosen to illustrate the procedure, the point H is chosen so that EH $= \lambda(\mathrm{H}) = 30°$, and the problem is to find the arc ΘH, or the declination δ(H) of H. The latter term is not used by Ptolemy, who speaks only of the arc between the equator and the ecliptic – another example of the scarcity of technical terms used in the Almagest.

As we saw in Chapter 3 (p. 75) Ptolemy solves such problems directly by the theorem of Menelaus, in the present case by applying it to the figure formed by one pair of great circles ZA and ZΘ intersected by the other pair EA and EB. But this procedure is equivalent to the application of formula (3.40) to the spherical triangle EΘH in which $\Theta = 90°$, $E = \varepsilon$, EH $= \lambda$(H) and ΘH $= \delta$(H). This gives

$$\sin \delta(\mathrm{H}) = \sin \lambda(\mathrm{H}) \cdot \sin \varepsilon \tag{4.2}$$

It is seen that if we define λ in the modern way as the distance of H from the vernal equinox we must substitute $\lambda + 180°$ for λ in (4.2). This entails a change of sign on the right hand side; but if we regard southern declinations as negative the relation still holds good.

The value $\delta(\mathrm{H}) = 30°$ was an arbitrary example. Actually, Ptolemy computes δ(H) for all integer values of λ(H) from 1° to 90° and tabulates the results in degrees, minutes, and seconds, in the so-called *Table of the Obliquity of the Ecliptic* [I, 15; Hei 1, 80].

Problem 2: To find the right ascension of a point on the ecliptic.

The next problem solved in the Almagest [I, 16; Hei 1, 82] is to determine the arc $E\Theta = \alpha(H)$ in Figure 4,1. Again, Ptolemy applies the theorem of Menelaos; the procedure leads to a relation which we can write in the form

$$\frac{\sin(90° - \varepsilon)}{\sin \varepsilon} = \frac{\sin(90° - \delta)}{\sin \delta} \cdot \sin \alpha$$

so that this problem, like the preceding one, can be solved by means of the table of chords. The relation is equivalent to the formula

$$\sin \alpha(H) = \tan \delta(H) \cdot \cot \varepsilon \tag{4.3}$$

which follows immediately from (3.41) applied to the rightangled spherical triangle $E\Theta H$.

The methods developed in Problems 1 and 2 enable Ptolemy to transform a pair of ecliptical co-ordinates (λ, β) into equatorial co-ordinates (α, δ) in the spècial case in which the ecliptic latitude $\beta = 0°$. The complete transformation as found in modern text books is given by the equations

$$\left.\begin{aligned} \sin \delta &= \sin \beta \cos \varepsilon + \cos \beta \sin \lambda \sin \varepsilon \\ \cos \delta \cdot \sin \alpha &= -\sin \beta \sin \varepsilon + \cos \beta \sin \lambda \cos \varepsilon \\ \cos \delta \cdot \cos \alpha &= \cos \beta \cos \lambda \end{aligned}\right\} \tag{4.4}$$

and the inverse transformation by

$$\left.\begin{aligned} \sin \beta &= \sin \delta \cos \varepsilon - \cos \delta \sin \alpha \sin \varepsilon \\ \cos \beta \cdot \sin \lambda &= \sin \delta \sin \varepsilon + \cos \delta \sin \alpha \cos \varepsilon \\ \cos \beta \cdot \cos \lambda &= \cos \delta \cos \alpha \end{aligned}\right\} \tag{4.5}$$

With $\beta = 0°$ these equations lead immediately to our formulae (4.2) and (4.3).

Problem 3: To transform ecliptical into equatorial coordinates.

How Ptolemy deals with the general case of coordinate transformation is seen in a later chapter of the Almagest [VIII, 5; Hei 2, 194] in which he considers a star Θ (see Figure 4.2) with given longitude and latitude and asks for its right ascension and declination. In the figure the pole of the equator is Z and that of the ecliptic is H. The great circle defined by Z and H has later been called the solstitial colure; in the Almagest it has no particular name. It is intersected by the equator in A and Γ, and by the ecliptic in B and Δ. E is the (vernal) equinox.

The great circle through H and Θ intersects the equator and the ecliptic at Λ and K respectively, while these are intersected in M and N by the great circle through Z and Θ. Then the star Θ has given

longitude $\lambda = EK$
latitude $\beta = K\Theta$

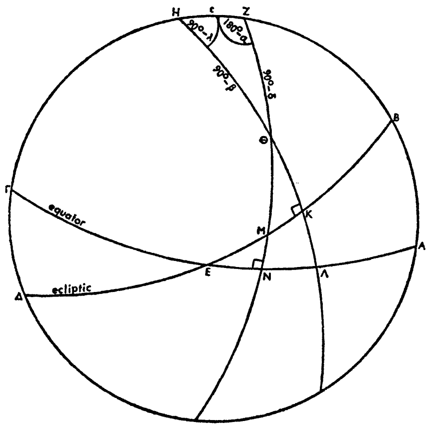


Fig. 4.2

relative to the ecliptic, and

right ascension $\alpha = \mathrm{EN}$
declination $\delta = \mathrm{N\Theta}$

relative to the equator. The latter are to be found.

To-day we would solve this problem by applying the spherical cosine formula to the general, polar triangle ΘHZ, giving

$$\cos(90° - \delta) = \cos\varepsilon \cdot \cos(90° - \beta) + \sin\varepsilon \cdot \sin(90° - \beta) \cdot \cos(90° - \lambda)$$

This formula and the corresponding procedure were unknown to Ptolemy, who regarded the figure as one pair of great circles through A (that is, AH and AN) intersected by another pair through Θ (ZN and HΛ). This enabled him to apply the theorem of Menelaus in the form (3.39). This gives, in the notation of Figure 4.2,

$$\frac{\sin \mathrm{HA}}{\sin \mathrm{AZ}} = \frac{\sin \mathrm{H\Lambda}}{\sin \Lambda\Theta} \cdot \frac{\sin \Theta\mathrm{N}}{\sin \mathrm{NZ}} \tag{4.6}$$

Here we have

$$\mathrm{HA} = 90° + \varepsilon \qquad \mathrm{NZ} = 90°$$
$$\mathrm{AZ} = 90° \qquad \mathrm{N\Theta} = \delta$$

whence (4.6) is reduced to

$$\cos\varepsilon = \frac{\sin \mathrm{H\Lambda}}{\sin \Lambda\Theta} \cdot \sin\delta \tag{4.7}$$

where

$$\sin \mathrm{H\Lambda} = \sin(90° + \Lambda\mathrm{K}) = \cos \Lambda\mathrm{K}$$

and

$$\sin \Lambda\Theta = \sin(\Lambda\mathrm{K} + \beta)$$

Here ΛK is not the declination of a point Λ on the ecliptic, since ΛK is not perpendicular to the equator. Nevertheless, it could be found from the table of the obliquity of the ecliptic [I, 15] if we 1) interchanged the equator and the ecliptic, and 2) knew the arc $E\Lambda$. The latter can be determined by a particular table [II, 8; Hei 1, 134] which we shall explain later (see Problem 8, page 110). Consequently, we realize that (4.7) enables us to find δ as a function of λ, β and ε, and that the procedure is equivalent to the first of the formulae in (4.4).

Ptolemy also applies the theorem of Menelaus in the form (3.36) This gives

$$\frac{\sin ZH}{\sin HA} = \frac{\sin Z\Theta}{\sin \Theta N} \cdot \frac{\sin N\Lambda}{\sin \Lambda A} \tag{4.8}$$

which for the given values of the variables is reduced to

$$\tan\varepsilon = \cot\delta \cdot \frac{\sin N\Lambda}{\cos \Lambda A}$$

Here we have $N\Lambda = NA + A\Lambda$
$ = 90^\circ - \alpha - \Lambda A$

where ΛA can be found from $KB = 90^\circ - \lambda$ by means of the table [II, 8] mentioned above. Thus (4.8) enables us to determine the right ascension α of the star. The method is an example of how Ptolemy proceeds when he is faced with a general spherical triangle.

The Concepts of Right and Oblique Ascension

To us the right ascension appears mainly as a co-ordinate which, together with the declination, determines the position of a point relative to a coordinate system based upon the celestial equator. In the preceding three problems we have regarded α and δ in this modern way. To Ptolemy – and to other early astronomers – it was different. Their basic system of coordinates was connected with the ecliptic, and the right ascension appeared not so much as a coordinate but as a characteristic function of importance to the theory of rising, setting, and culmination times. In fact two stars Θ_1 and Θ_2 with right ascensions α, and $\alpha_2 > \alpha_1$, will cross the meridian (i.e. culminate) at times t_1 and t_2 such that

$$\alpha_2 - \alpha_1 = (t_2 - t_1) \cdot 15^\circ \tag{4.9}$$

if α is measured in degrees and t in hours. Therefore, two stars with the same right ascension will culminate at the same moment. If their right ascensions are different, $(\alpha_2 - \alpha_1)$ may be used as a measure of the difference of the culmination times – either so that right ascensions are expressed directly in hours and minutes of time as in modern astronomy, or so that they are given in 'degrees of time' as in Ptolemy, one degree of time being 1/360 of the duration of one complete revolution of the

heavens. In the Almagest the first Book concludes with a calculation based on (4.3) and showing the degrees of time corresponding to the first three signs of the ecliptic, viz.

Aries.....	27°;50
Taurus...	29°;54
Gemini...	32°;16

By an approximation method Ptolemy can then construct a small table [I, 17; Hei 1,85] giving the degrees of time corresponding to 10°-intervals of the three signs.

Culmination times are important to modern astronomy due to the use of the meridian circle as the most precise instrument of measurement. Greek astronomers did, in fact, measure in the meridian (with cruder instruments); but in early astronomy much more importance was given to measurements at the horizon, i.e. to determinations of the points and times at which a star rose above the horizon, or set below it. Here no particular instruments were needed, apart from some kind of clock. The trouble was that whereas differences in culmination times were the same as differences in right ascensions, the latter are not connected with rising times in any simple way.

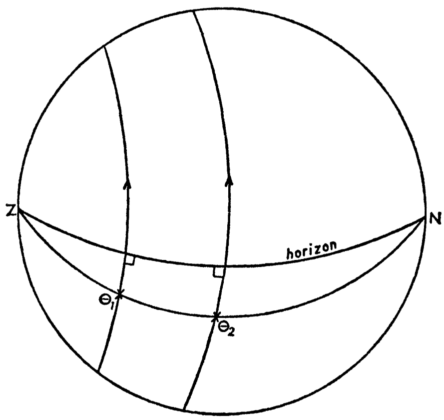


Fig. 4.3

In general two stars with the same right ascension $\alpha_1 = \alpha_2$ will culminate at the same time, but rise at different times – except for one particularly privileged class of observers, viz. those who regard the heavens from some locality upon the terrestrial equator. At such a place the axis of the world with its two poles N and Z (see Figure 4.3) will lie in the plane of the horizon. Therefore all the points of a declination circle $N\Theta_2\Theta_1Z$ from pole to pole will rise at the same moment, and any point on it will move upon a circle forming a right angle with the horizon. This is expressed by saying that an observer at the equator views the heavenly sphere as a *sphaera recta*, and that

all the stars to him have a right ascension in the literal meaning of the words. Speaking of right ascension in the modern sense it is obvious that 1) two stars having the same right ascension will rise at the same time, and 2) two stars with different right ascensions will rise at different times in accordance with (4.9).

The observer on the equator is privileged in so far as he is able to determine differences of rising times as differences of right ascensions, and vice versa. But his location on the Earth is a very special one, wherefore spherical astronomy must pay more attention to observers elsewhere who view the sphere as a *sphaera obliqua* with a polar altitude equal to the geographical latitude φ. This general case is the subject matter of the second Book of the Almagest, which begins with the curious statement that the Earth is 'at present' inhabited only upon one half of the northern hemisphere [II, 1; Hei 1, 88]. This is founded upon purely astronomical reasons. Thus Ptolemy states that he has never heard of gnomons with shadows pointing towards the South at noon on the date of an equinox. Therefore all observers – that is, all observers known to Ptolemy – must be situated on the Northern hemisphere. Neither has he had any report that the same eclipse has been observed by two different observers with a time interval greater than 12^h. Accordingly, observers must live upon one half of the northern hemisphere only.

B: Astronomical Geography

The problems 1–3 described above were related to transformations of coordinates; in a way they belong to spherical trigonometry as such rather than to spherical astronomy. The latter is the subject matter of Book II of the Almagest which (after an introductory chapter) opens with a series of problems and procedures related to the motion of the Sun as seen by observers at different times of the year, or at different places of the Earth. They all suppose the observer to be placed in the *sphaera obliqua*, North of the equator, and it will be seen that most of the problems are connected with observations at the horizon.

Problem 4: To find the amplitude of the Sun.

In the morning the Sun rises at a point on the eastern horizon. The angular distance of this point from the East is nowadays called the amplitude. This term is not found in Ptolemy, who refers to the amplitude as the arc of the horizon between the equator and the ecliptic. [II, 2; Hei 1, 89].

Figure 4.4 shows the meridian with the eastern horizon ΒΕΔ and the equator ΑΕΓ. The Sun is at the point Η which is just rising above the horizon. The great circle ΝΗΖ cuts the equator in Θ so that $H\Theta = \delta(H)$ is the declination of the Sun, and $a = HE$ the amplitude to be determined. Since the points Η (regarded as a fixed point on the ecliptic) and Θ will cross the meridian at the same time, the arc ΘΑ will measure the time between sunrise and noon. Therefore the angle ΑΝΘ will be $t/2$ where t is the

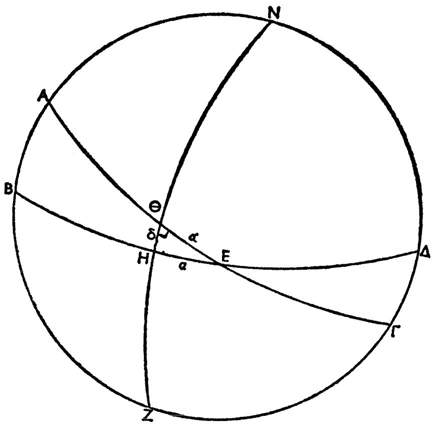


Fig. 4.4

length of the natural day corresponding to the declination $\delta(H)$. By means of (3.40) applied to the spherical triangle NBH with a right angle at B we find

$$\sin BH = \sin HN \cdot \sin t/2$$

or

$$\sin (90° - a) = \sin (90° + \delta) \cdot \sin t/2$$

or, finally

$$\cos a = \cos \delta \cdot \sin t/2 \qquad (4.10)$$

Ptolemy illustrates the procedure by means of a typical example, computing the amplitude at winter solstice at Rhodes where the longest day is $14\frac{1}{2}^h$, and the shortest day $24^h - 14\frac{1}{2}^h = 9\frac{1}{2}^h$. At winter solstice the declination of the Sun is $\delta = -\varepsilon = -23°;51,20$. The relation (4.10) gives in this case $a = 30°;0$, in agreement with Ptolemy's result. The Sun will then rise 30° South of East. As far as we know Ptolemy never observed at Rhodes. The example is, perhaps, taken from Hipparchus who worked there during at least a part of his life, but it could also be that it was chosen because Rhodes served as the origin of a system of geographical coordinates introduced by Eratosthenes[2]) and was thus a well known locality.

It is worth noticing that Ptolemy does not take atmospheric refraction into account (cf. page 42). This means that the amplitude computed from (4.10) will be wrong. Actually the refraction has the effect that the visible or apparent image of the Sun is lifted above the position determined by its material centre. The Sun will thus rise before it has reached the amplitude given by (4.10).

2) Eratosthenes based his system upon two standard lines. The first passed from the Pillars of Hercules, through Rhodes to the Gulf of Issus and was supposed to be a circle parallel to the equator. The second passed through Byzantium, Rhodes, Alexandria, and the Nile island of Meroë and was regarded as part of a meridian circle. Cf. Strabo, *Geogr.* II, 1,1 and I,4, 1–2.

Problem 5: To find the latitude from the length of daylight.

In the previous problem formula (4.10) enabled us to determine the amplitude of the Sun as a function of its declination δ, which varies throughout the year, and of of the length t of the day from sunrise to sunset. Now t is itself a function of two variables, viz. δ and the geographical latitude φ of the observer. This latter relation can be examined by Figure 4.4, where the pole N is assumed to be φ degrees above the northern horizon, the geographical latitude φ being equal to the altitude of the pole. Again we consider the spherical triangle NBH, but this time we apply (3.41), with the result that

$$\tan (90^\circ - a) = \sin (180^\circ - \varphi) \cdot \tan t/2$$

or

$$\cot a = \sin \varphi \cdot \tan t/2 \tag{4.11}$$

The equivalent of this formula is found by Ptolemy by means of the theorem of Menelaus [II, 3; Hei 1, 93]. If we write (4.11) as

$$\sin \varphi = \cot a \cdot \cot t/2 \tag{4.12}$$

we see that the geographical latitude can be found from measurements of the amplitude and the length of daylight.

Eliminating a from (4.10) and (4.11) we find

$$\sin \varphi = \pm \frac{\cos \delta \cdot \cos t/2}{\sqrt{1 - \cos^2 \delta \cdot \sin^2 t/2}} \tag{4.13}$$

which gives the geographical latitude as a function of the length of the day and the declination of the Sun. Since δ is given by (4.2), we see that it is necessary to know the ecliptic longitude λ of the Sun at the date in question. But λ varies throughout the year in a rather complicated way described in Book III of the Almagest; therefore, (4.13) is useless until we have developed a theory of the motion of the Sun, giving λ as a function of time.

However, in one particular case (4.13) can be used without any detailed knowledge of the theory of the Sun. We know that the longest day $t_{max} = T$ occurs when the Sun has its maximum declination $\delta_{max} = \varepsilon$. In this case (4.13) gives

$$\sin \varphi = \pm \frac{\cos \varepsilon \cdot \cos T/2}{\sqrt{1 - \cos^2 \varepsilon \cdot \sin^2 T/2}} \tag{4.14}$$

which shows that the geographical latitude can be found directly from the length of the longest day.

Chapter II, 3 ends with two corollaries which follow easily from Figure 4.5. Here Δ and B are the solstitial and ♈ and E the equinoctial points of the ecliptic. The positions S_1 and S_2 are on the same parallel, and thus have the same distance from Δ, while the points S_3 and S_4 have the same distances from the equinoxes as S_1 and S_2 respectively. The two corollaries can then be stated as a single proposition, viz.

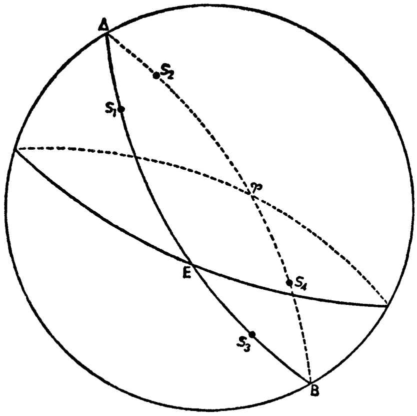


Fig. 4.5

If S_1, S_2, S_3, and S_4 are positions of the Sun, the length of the day will be the same at S_1 and S_2, and equal to the length of the night at S_3 and S_4.

Problem 6: To find when the Sun is at zenith.

This problem is dealt with in a very short chapter where Ptolemy outlines only his method, giving no actual calculations [II, 4; Hei 1, 97].

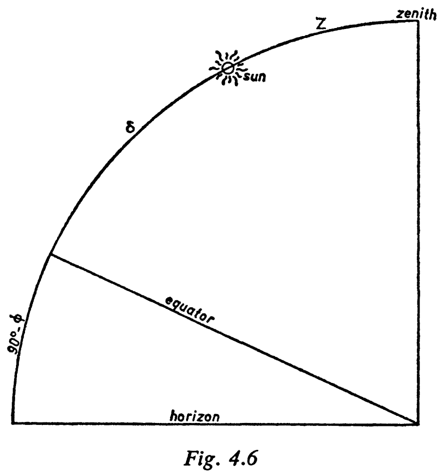


Fig. 4.6

Let us consider (Figure 4.6) a date when the Sun has the declination δ at the moment when it is passing the meridian in upper culmination. At this moment its zenith distance will be

$$z = \varphi - \delta \tag{4.15}$$

which is seen at once from Figure 4.6. If the Sun is at the zenith when passing the meridian we must have $z = 0$, i.e. its declination $\delta = \varphi$ must be equal to the geographical latitude. This means that the Sun will never be at zenith North of the tropic of Cancer (or South of the tropic of Capricorn), since the maximum value of the declination is $\delta_{max} = \varepsilon$. Between the two tropics we have $|\varphi| < \varepsilon$ and the Sun will be at zenith twice a year. The equation $\delta(\lambda) = \varphi$ will have two solutions, $\lambda_1 = \lambda(t_1)$ and $\lambda_2 = \lambda(t_2)$, corresponding to the dates t_1 and t_2. As in the preceding problem these dates cannot be found without theoretical knowledge of the motion of the Sun.

Problem 7: To find the length of shadows.

This problem is very simple and involves only plane trigonometry [II, 5; Hei 1, 98]. It gives the theory of the simplest and most ancient of astronomical instruments, the gnomon, which is nothing but a pole or stick raised perpendicularly to a horizontal plane surface, on which it will cast a shadow when the Sun is out. From the direction and length of this shadow a surprisingly great amount of information on the daily and annual motion of the Sun can be acquired[3]).

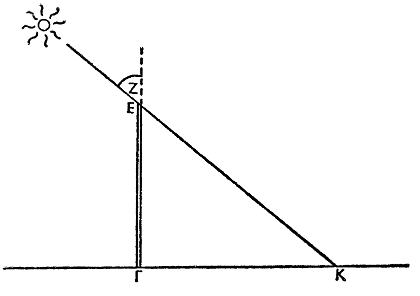


Fig. 4.7

Here we consider (Figure 4.7) the shadow cast by the Sun of a gnomon ΓE the length of which can be chosen arbitrarily as 60^p. It is seen from (3.25) that the length of the shadow is given by

$$s = \Gamma K = 60^p \cdot \tan z \tag{4.16}$$

where z is the zenith distance of the Sun. Because his only trigonometrical equipment is the table of chords, Ptolemy is unable to handle an expression like (4.16) and has to proceed in a way which is very characteristic of his computational technique in plane trigonometry. The procedure has been explained in Chapter 3 (page 66) where (3.25) was derived.

3) There is every reason to consider the gnomon the earliest astronomical instrument. It was presumably known in prehistoric times. Herodotus (II, 109) asserts that the Greeks learned of its use from the Babylonians, while Diogenes Laërtius (II,1) wrongly named Anaximander as its inventor. – Cfr. Dicks 1954 p. 77.

It appears that Ptolemy is particularly interested in the length of the shadow at noon at the beginning of the four seasons, where we have at the

$$\begin{array}{lll} \text{equinoxes} \ldots\ldots\ldots\ldots & s_0 = 60^p \cdot \tan \varphi & \\ \text{summer solstice} \ldots\ldots & s_1 = 60^p \cdot \tan (\varphi - \varepsilon) & (4.17) \\ \text{winter solstice} \ldots\ldots & s_2 = 60^p \cdot \tan (\varphi + \varepsilon) & \end{array}$$

If we can measure two of the three quantities s_0, s_1, and s_2 it is possible to determine both the latitude and the obliquity of the ecliptic, but Ptolemy's concluding remark [II, 5; Hei 1, 101] is that this method is less exact since it is difficult to determine the length of the shadow at winter solstice, and also the time of the equinoxes. In view of this remark one could ask why Ptolemy found it necessary to include the theory of the gnomon in the Almagest. The reason becomes clear in the following, where both the theory of the gnomon, the zenith position of the Sun, and the theory of the longest day of the year are used as a basis for the ancient division of the Earth into geographical zones.

The Division of the Earth into Zones

The three previous problems have prepared the way for a special chapter [II, 6; Hei 1, 101] in which Ptolemy gives the astronomical theory of the division of the Earth into geographical zones between circles parallel to the terrestrial equator. He begins by considering an observer placed on the equator itself. As we have seen (page 100) such an observer is privileged in more ways than one:

1) To him all the stars of both the northern and the southern heavens are visible.
2) He sees the heavenly sphere as a *sphaera recta* with all diurnal arcs perpendicular to the horizon.
3) Throughout the year day and night have an equal length of 12^h.

Because of the last point the concept of the equinoxes loses its usual meaning. The equinoxes have to be redefined as the two dates upon which the Sun culminates at zenith. On these dates a gnomon will cast no shadow at noon. It is also impossible to define the Summer and Winter seasons by the length of the day; but the equinoctial dates divide the year into two periods, during one of which the noon shadow of a gnomon will point to the South, and during the other to the North, according to whether the declination of the Sun is positive (northern) or negative (southern).

Let us now consider an observer outside the equator and situated at a place with geographical latitude φ in the northern hemisphere (Ptolemy is never interested in how phenomena would appear to a possible observer in the southern hemisphere).

1) To such an observer a certain part of the heavens around the southern pole is invisible forever, while a corresponding region around the northern pole is always visible; the stars of this latter region never rise or set, and are said to be circumpolar.

2) The celestial sphere is seen as a *sphaera obliqua*, and all non-circumpolar stars rise and set at oblique angles with the horizon.

3) Except at the equinoxes the lengths of day and night are different, that of the longest day (summer solstice) being determined indirectly by (4.14).

The variation of the longest day T with the geographical latitude φ is now taken as the basis of a division of the surface of the Earth into 38 zones by means of 39 circles parallel to the equator, the first ($n = 1$) being the equator itself, while the last ($n = 39$) has shrunk to a point identical with the North pole. The principle is the following: a round value T_n of T is substituted for T in (4.14) and the corresponding latitude φ_n is calculated. Then (4.17) gives the lengths of the noon shadows of a gnomon of length 60^p at the equinoxes (s_0), summer solstice (s_1) and winter solstice (s_2). As an example, we have for $n = 2$ the latitude

$$\varphi_2 = 4°;15$$

and the shadows

$$s_{02} = 4^p;25$$
$$s_{12} = 21^p;20$$
$$s_{22} = 32^p;0$$

The corresponding parallel circle is said to pass through the Island of Taprobane, which has been identified with Ceylon and is used as the standard locality by which the second parallel is named.

All the quantities computed in this way are collected in the following *Table of Parallels*. Some of the standard localities mentioned in this table are difficult to identify. There is no doubt that *Caturactionum* of Britain ($n = 24$) is Catterick in Yorkshire, and that *Brigantium* ($n = 22$) is the territory of the tribe of the *Brigantes*, extending from York to Hadrian's Wall. But the usual identification of the *Mouth of Tanais* ($n = 21$) with the mouth of the Don (latitude 47°;12) agrees badly with the latitude given in the table. A more probable guess is to identify this place with the Roman *Tanaus*, which is perhaps the mouth of the river Tees (latitude 54°;37.)

Table of Parallels

n	T	φ_n	s_0	s_1	s_2	Standard locality
1	12	0	0	26;30	26;30	The Equator
2	$12\frac{1}{4}$	4;15	4;25	21;20	32;0	The Island of Taprobane (Ceylon)
3	$12\frac{1}{2}$	8;25	8;50	16;35	37;54	The Aualitic Gulf (Gulf of Aden)
4	$12\frac{3}{4}$	12;30	13;20	12;0	44;10	The Adulitic Gulf (Annesley Bay)
5	13	16;27	17;45	7;45	51;0	The Island of Meroë
6	$13\frac{1}{4}$	20;14	22;10	3;45	58;10	The Napatian Country
7	$13\frac{1}{2}$	23;51	26;30	0;0	65;50	Syëne (Assuan)
8	$13\frac{3}{4}$	27;12	30;50	3;30	74;10	Ptolemais Hermeïon
9	14	30;22	35;5	6;50	83;12	Lower Egypt
10	$14\frac{1}{4}$	33;18	39;30	10;0	93;5	The Middle of Phoenicia
11	$14\frac{1}{2}$	36;0	43;36	12;55	103;20	Rhodes
12	$14\frac{3}{4}$	38;35	47;50	15;40	114;55	Smyrna
13	15	40;56	52;10	18;30	127;50	The Hellespont
14	$15\frac{1}{4}$	43;1	55;55	20;50	140;15	Massalia (Marseille)
15	$15\frac{1}{2}$	45;1	60;0	23;15	155;5	The Middle of Pontus
16	$15\frac{3}{4}$	46;51	63;55	25;30	171;10	The Source of Ister (The Danube)
17	16	48;32	67;50	27;30	188;35	The Mouth of Borysthenes (Dnieper)
18	$16\frac{1}{4}$	50;4	71;40	29;55	208;20	The Palus Maeotis (The Sea of Azov)
19	$16\frac{1}{2}$	51;30	75;25	31;25	229;20	Southern Britain
20	$16\frac{3}{4}$	52;50	79;5	33;20	253;10	The Mouth of Rhenus (The Rhine)
21	17	54;1	82;35	34;55	278;45	The Mouth of Tanaïs
22	$17\frac{1}{4}$	55;0	85;40	36;15	304;30	Brigantium
23	$17\frac{1}{2}$	56;0	88;50	37;40	335;15	The Middle of Great Britain
24	$17\frac{3}{4}$	57;0	92;25	39;10	372;40	Caturactionum of Britain
25	18	58;0	96;0	40;40	419;5	Southern Part of Little Britain
26	$18\frac{1}{2}$	59;30				The Middle of Little Britain
27	19	61;0				The Northern Part of Little Britain
28	$19\frac{1}{2}$	62;0				The Ebudic Islands (The Hebrides)
29	20	63;0				The Island of Thule
30	21	64;30				Unknown Scythian Nations
31	22	65;30				
32	23	66				
33	24	66;8,40				
34	1 month	67				
35	2 months	69;30				
36	3 –	73;20				
37	4 –	78;20				
38	5 –	84				
39	6 –	90				

We notice that the 7th parallel is the same as the Tropic of Cancer, upon which the geographical latitude $\varphi = 23°;51$ is equal to the obliquity ε of the ecliptic given by (4.1). There the Sun is at zenith once a year. South of this parallel the noon shadow of a gnomon may point either to the North or to the South, depending on the time of

the year. Therefore parallels Nos 1–6 are called amphiscian (*amphiskios* = throwing shadows both ways) while No 7 is the first of the heteroscian parallels (*heteroskios* = throwing shadows one way only).

The first 25 parallels are defined by time intervals of 15 minutes. Since their relative distance is a decreasing function of latitude, Ptolemy now increases the time interval, first to 30 minutes and next to 1 hour, until we reach parallel No 34. This is the first of the periscian[4]) parallels (*periskios* = throwing shadows all the way round) on which the lower culmination of the Sun is visible (the Arctic circle). It has the peculiar property that here the ecliptic coincides with the horizon at the vernal and autumnal equinoxes. Further towards the North Ptolemy restricts himself to an approximate calculation of the parallels where the longest day is 1, 2, 3, 4, 5, and 6 months respectively.

The eleventh parallel is particularly important to Hellenistic astronomy since it passes through Rhodes where Hipparchus made most of his observations. It goes also through Cnidos where according to Strabo (*Geogr.* II, 5, 14) Eudoxos had his observatory. As already mentioned Ptolemy observed in Alexandria, but the precise location of his observatory is unknown. In the Almagest [V, 12; Hei 1, 407] the latitude of Alexandria is given as 30°;58 which places the city between the ninth and the tenth parallel. Modern measurements show Alexandria to have a latitude of about 31°;12 so that Ptolemy's value is one quarter of a degree too small[5]).

The subdivision of the oecumene, or habitable part of the world, into zones or climates parallel to the equator was a question which occupied Greek geographers at a very early date, and the names of Parmenides, Aristotle, Eratosthenes and others were associated with it. Eratosthenes was criticized by Hipparchus for relying too much on dubious accounts by travellers, and too little on precise astronomical observations[6]). Ptolemy's definition of the parallels by astronomical and gnomonical methods was the most detailed treatment of the problem on a mathematical basis; here as in many other places he sought to carry through a programme to which his favourite master Hipparchos had called attention. In his *Geography* the division into zones was made the foundation of a detailed geographical description of the Earth[7]). In the Almagest Ptolemy had a more astronomical purpose in mind. Here the division served as a framework for characterizing the position of an observer. This will become more evident in the following sections.

4) According to Strabo (*Geogr.* II, 2, 3) the terms *amphiscian*, *heteroscian*, and *periscian* were coined by Posidonius (about 100 B.C.).

5) Peters and Knobel (1915, p. 14) ascribed the discovery of this error to Delambre (1817, II, p. 211 and 284). In fact, it was known to previous astronomers, such as Roger Long (1742, p. 279).

6) See Honigman (1929) and the bibliography in Brown (1949). For Parmenides, see Strabo (*Geogr.* II, 2,2), and for Aristotle the *Meteor.* (II, 5, 362 a). Hipparchus' criticism of Eratosthenes is mentioned in Strabo (*Geogr.* I, 1, 12).

7) Ptolemy's *Geography* remained unknown to the Latin Middle ages until about A.D. 1410 when a Byzantine scholar Emanuel Chrysoloras and his Italian pupil Jacobus Angelus produced a translation. This became the basis of the first printed version (Vincenza, 1475). Since then the *Geography* has appeared in more editions and translations than any of Ptolemy's other works (see Sarton, 1927, I, 273 ff.). For a brief survey of its contents, see Brown 1949, pp. 58–80.

C: Rising and Setting of the Stars

Problem 8: To find times of rising in the *sphaera obliqua*.

Having dealt with the geographical parallels, Ptolemy is now prepared to tackle more composite spherical phenomena as they appear to observers under different latitudes. First he returns to the problem of times of rising. We saw above (page 99) how he was able to calculate the time used by an arc on the ecliptic to rise above the horizon in the *sphaera recta*, that is for an observer placed on the terrestrial equator. This was a simple problem since the rising time of the arc is equal to the difference of the right ascensions of its end points (cf. 4.9) provided that right ascensions are expressed in hours and minutes. We shall now consider the more complicated problem of determining such rising times in the *sphaera obliqua* where (4.9) does not obtain, since the diurnal arc of any celestial object is no longer perpendicular to the horizon of the observer, who is supposed to have the geographical latitude φ [II, 7; Hei 1, 117].

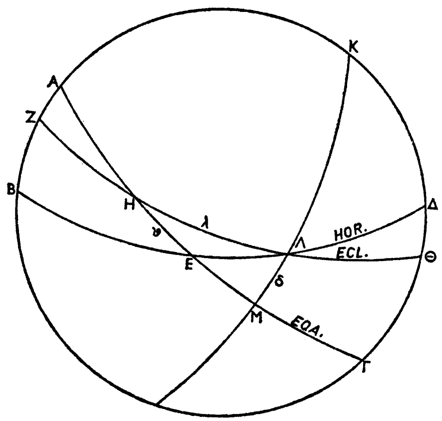


Fig. 4.8

Figure 4.8 shows the meridian ABΓΔ, the equator AEΓ through the East point E of the horizon BEΔ, and the ecliptic ZHΘ. The figure is so drawn that H is the vernal equinox, and K the northern pole of the equator. The point Λ of the ecliptic with the longitude $\lambda(\Lambda) = \mathrm{H}\Lambda$ is just rising above the horizon together with the point E of the equator. The problem is to find the arc $\vartheta = \mathrm{HE}$ on the equator for a given value of λ.

As usual the solution is based on the theorem of Menelaus, this time applied to the figure formed by the arcs ΓE and ΓK intersected by EΔ and KΛ. The method is explained in the form of a numerical example in which $\lambda(\Lambda) = 30°$ so that the arc HΛ is identical with the sign of Aries. We shall not follow this computation in detail but extract the general method from the example.

The great circle KΛ intersects the equator in M. The arc to be found is

$$\vartheta = \mathrm{HE} = \mathrm{HM} - \mathrm{EM} = \alpha(\Lambda) - \mathrm{EM} \qquad (4.18)$$

where $\alpha(\Lambda)$ is the right ascension of the rising point Λ. It can be expressed by (4.3) as

$$\sin \alpha(\Lambda) = \tan \delta(\Lambda) \cdot \cot \varepsilon \tag{4.19}$$

where $\delta(\Lambda)$ is the declination of Λ determined by (4.2), or by

$$\sin \delta(\Lambda) = \sin \lambda(\Lambda) \cdot \sin \varepsilon \tag{4.20}$$

The triangle $E\Lambda M$ has a right angle at M while the angle E is measured by the arc $\Delta\Gamma$ and therefore equal to $(90° - \varphi)$. By (3.41) we then have

$$\tan \delta(\Lambda) = \sin EM \cdot \tan (90° - \varphi)$$

or

$$\sin EM = \tan \delta(\Lambda) \cdot \tan \varphi \tag{4.21}$$

It follows from (4.21) that the problem has no solution if

$$\tan \delta \cdot \tan \varphi > 1$$

that is if

$$\delta > 90° - \varphi$$

that is if the point Λ of the ecliptic is circumpolar at the given latitude.

Eliminating α and δ we can write (4.18) in the form

$$\vartheta(\lambda, \varphi) = \text{arc sin} \left\{ \frac{\sin \lambda \cdot \cos \varepsilon}{\sqrt{1 - \sin^2 \lambda \cdot \sin^2 \varepsilon}} \right\} - \text{arc sin} \left\{ \frac{\sin \lambda \cdot \sin \varepsilon \cdot \tan \varphi}{\sqrt{1 - \sin^2 \lambda \cdot \sin^2 \varepsilon}} \right\} \tag{4.22}$$

or

$$\sin \vartheta(\lambda, \varphi) = \frac{\sin \lambda [\cos \varepsilon \sqrt{1 - \sin^2 \lambda \cdot \sin^2 \varepsilon (1 + \tan^2 \varphi)} - \sin \varepsilon \cdot \cos \lambda \cdot \tan \varphi]}{1 - \sin^2 \lambda \sin^2 \varepsilon} \tag{4.23}$$

which gives the rising time of the whole arc $H\Lambda$ as a function of λ and φ as the only independent variables. It is seen that they are both 'strong' in the sense defined above (page 86). Thus $\vartheta(\lambda, \varphi)$ must be represented by a table with a double entry.

This *Table of Oblique Ascensions* [II, 8; Hei 1, 134] is arranged as a system of 11 partial tables, each of which gives $\vartheta(\lambda, \varphi_n)$ as a function of λ only with a constant value φ_n of the latitude. For φ_n Ptolemy chooses the values determined by $n = 1, 3, 5 \ldots 21$ in the table of parallels page 108, that is, he selects the parallels upon which the longest day is 12^h, $12^h;30$, 13^h ... 17^h. Each of the partial tables has 36 rows and 3 columns.

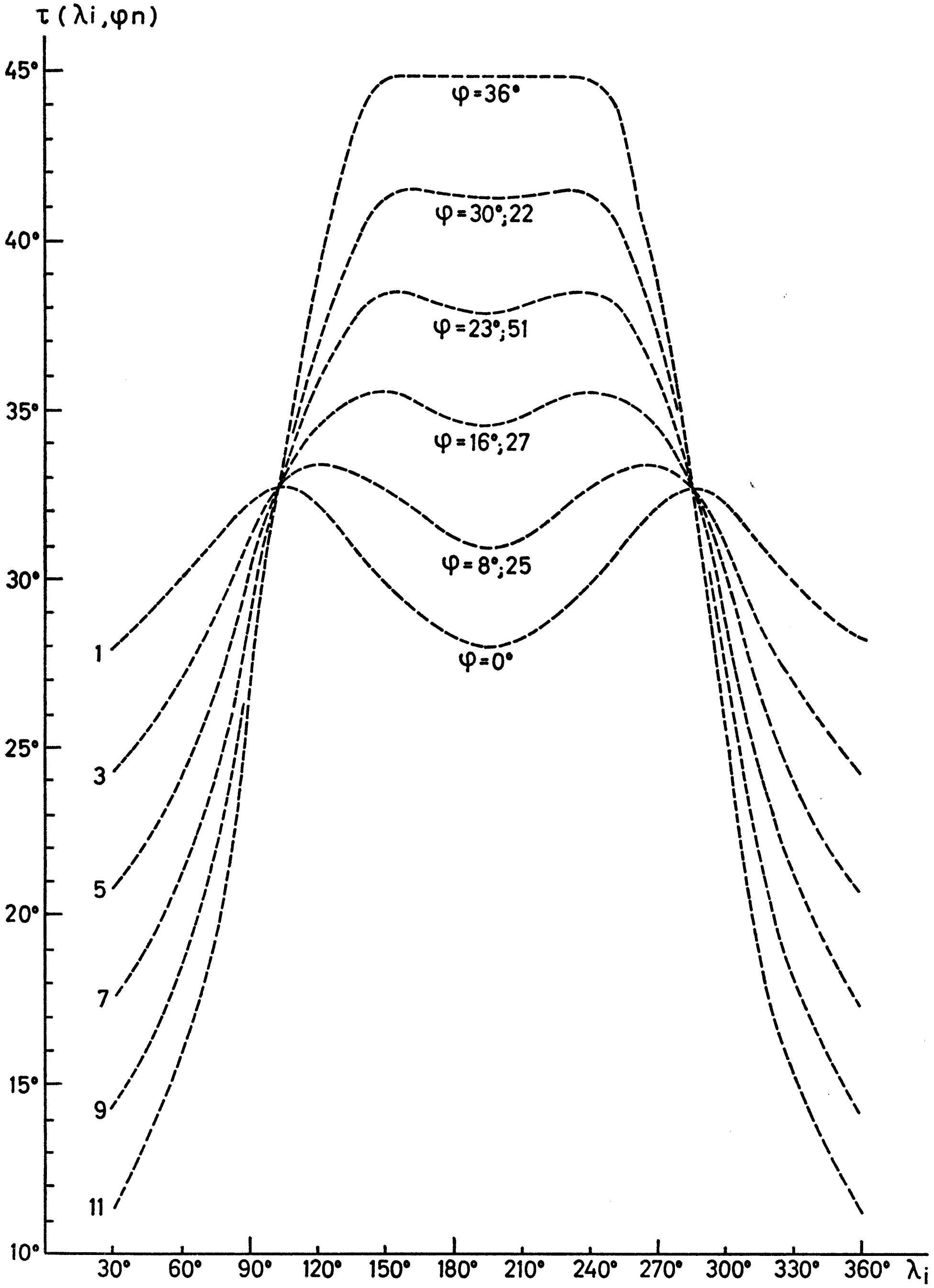
Column 1 contains the argument λ_i $(i = 1, 2, \ldots 36)$ reckoned in signs and degrees from the vernal equinox with intervals of 10° from $\lambda_1 = 10°$ to $\lambda_{36} = 360°$.

Column 3 contains $\vartheta(\lambda_i, \vartheta_n)$, that is the rising time of the whole arc λ_i (reckoned from the vernal equinox) on the parallel with the latitude φ_n.

Column 2 lists the difference

$$\tau(\lambda_i, \varphi_n) = \vartheta(\lambda_i, \varphi_n) - \vartheta(\lambda_{i-1}, \varphi_n) \tag{4.24}$$

which for $\lambda_i = 0°, 10°, 20°, \ldots 360°$ represents the rising time of each 10°-interval on the ecliptic.



τ (λi, φn)
45°
40°
35°
30°
25°
20°
15°
10°
φ = 36°
φ = 30°; 22
φ = 23°; 51
φ = 16°; 27
φ = 8°; 25
φ = 0°
1
3
5
7
9
11
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
360°
λi

Fig. 4.9

The similar function

$$\tau(\lambda_i, \varphi_n) = \vartheta(\lambda_i, \varphi_n) - \vartheta(\lambda_i - 30^\circ, \varphi_n) \tag{4.25}$$

represents (for $\lambda_i = 30°, 60°, \ldots 360°$) rising times of 30° intervals of the ecliptic in a number of climates. It is not tabulated in the Almagest, but shown in Figure 4.9. It is seen that for small latitudes there is no great variation of the rising times of the various signs, which here rise together with arcs of about 30° of the equator. At greater latitudes the deviations are more impressive. Thus at $\varphi_{21} = 54°;30$ the sign of Aries has a rising time of 11°;19 degrees of time, while for Leo it is 44°;22.

It appears from the diagram and is easily shown from (4.22) and (4.25) that the curves have two points in common, corresponding to $\lambda = 105°$ and 285°. Here the rising time of an interval on the ecliptic is independent of the latitude of the observer.

The Use of the Table of Oblique Ascensions

The Table of Oblique Ascensions is one of the most versatile and powerful tools of Ptolemy's spherical astronomy. As we are going to see, it enables us to solve a number of problems which otherwise would have implied rather cumbersome trigonometrical calculations.

Of particular interest is column 3 of the first partial table, listing $\vartheta(\lambda_i, 0°)$ corresponding to $\varphi = \varphi_1 = 0°$, i.e. the equator. Since we here have $\vartheta(\lambda_i, 0°) = \alpha(\lambda_i)$ (see page 101) this first table can serve as a *table of right ascensions* of points on the ecliptic. Comprising the whole ecliptic, it is, of course, more convenient than the previous table of right ascensions of the first 3 signs (page 100). Here we are reminded once more that right ascensions are not conceived as co-ordinates, but as a particular set of rising times, viz. those valid in the *sphaera recta.*

Other applications are directly concerned with time and its measurement; they are very briefly outlined by Ptolemy in a particular chapter [II, 9; Hei 1, 142]. Here we are taught, first, how the length t of the artificial day can be found in a simpler manner than by (4.11) as

$$t = \vartheta(\lambda_\odot + 180°, \varphi) - \vartheta(\lambda_\odot, \varphi) \tag{4.26}$$

where $\lambda_\odot$ is the longitude of the Sun, and t is expressed in degrees of time. No proof is given. It follows from Figure 4.8, where Λ now represents the rising Sun, and the point on the ecliptic opposite to Λ corresponds to the point on the equator setting together with the Sun.

At this place Ptolemy introduces the definitions of the two kinds of hour used in Antiquity, viz. the equinoctial and the seasonal hours.

The equinoctial hours are equal both day and night all the year round. One such hour corresponds to 15 degrees of time (cf. 4.9), so that the length of the day in equinoctial hours is obtained dividing (4.26) by 15°.

Seasonal hours are of different length during day and night, and also during the year. One seasonal hour of the day is defined as one twelfth of the length of daylight as given by (4.26). It has a maximum length at summer solstice and a minimum at winter solstice; at the equinoxes seasonal hours are the same as equinoctial hours, and this always is the case, too, on the equator. Similarly one twelfth of the length of the night is one nocturnal hour; from the vernal to the autumnal equinox one nocturnal hour is shorter than one hour of the day, but during the rest of the year it is longer.

It follows that we can transform equinoctial hours into seasonal hours, and vice versa, by the relation

$$12 \text{ seas. hours} = \frac{\vartheta(\lambda_{\odot} + 180°, \varphi) - \vartheta(\lambda_{\odot}, \varphi)}{15°} \text{ equin. hours} \tag{4.27}$$

Ptolemy also states another rule based on the fact that when one moves from the equator to a northern parallel of the latitude φ the rising time of the Sun will decrease from $\vartheta(\lambda_{\odot}, 0°)$ to $\vartheta(\lambda_{\odot}, \varphi)$. Therefore half the day will increase by the amount $\vartheta(\lambda_{\odot}, 0°) - \vartheta(\lambda_{\odot}, \varphi)$. Distributing this difference equally over 6 hours we get the length of one seasonal hour in degrees of time as

$$1 \text{ seas. hour} = 15° + \frac{\vartheta(\lambda_{\odot}, 0°) - \vartheta(\lambda_{\odot}, \varphi)}{6} \tag{4.28}$$

We are now prepared to find the longitude λ_h of the ascendant, or the horoscope, that is the rising point of the ecliptic, at a time given as t equinoctial hours after sunrise at a date when the Sun has the longitude $\lambda_{\odot}$. At this point of time an arc of $t \cdot 15°$ has risen above the horizon after the Sun. Adding the oblique ascension $\vartheta(\lambda_{\odot}, \varphi)$ of the Sun taken from the table of oblique ascensions, we have

$$t \cdot 15° + \vartheta(\lambda_{\odot}, \varphi)$$

as the right ascension of the point of the equator rising at the time t; but this is the oblique ascension of the rising point of the ecliptic, whence

$$\vartheta(\lambda_h, \varphi) = t \cdot 15° + \vartheta(\lambda_{\odot}, \varphi) \tag{4.29}$$

from which λ_h can be found by the table of oblique ascensions.

Finally it is possible to find the longitude λ_e of the culminating point of the ecliptic – called by Mediaeval Latin astronomers, the *medium coeli*, or mid-heaven – at a time given as t equinoctial hours after true noon, that is after the Sun has passed the meridian. At this time an arc of the equator equal to $t \cdot 15°$ has crossed the meridian after the Sun. Adding the right ascension $\vartheta(\lambda_{\odot}, 0°)$ of the Sun taken from the table of oblique ascensions for the *sphaera recta* we have

$$t \cdot 15° + \vartheta(\lambda_{\odot}, 0°)$$

as the right ascension of the culminating point of the equator. This is the right

ascension $\vartheta(\lambda_e, 0°)$ of the culminating point of the ecliptic, so that we have the relation

$$\vartheta(\lambda_e, 0°) = t \cdot 15° + \vartheta(\lambda_\odot, 0°) \tag{4.30}$$

enabling us to find λ_e by the table of right ascensions. Note that the Ptolemaic procedures corresponding to (4.29) and (4.30) are slightly different, using seasonal hours.

D: The Instantaneous Position of the Ecliptic

In Ptolemy's theoretical astronomy the most important great circle is the ecliptic to which the movements of the Sun, Moon and planets are referred. Since the ecliptic is an invisible circle participating in the diurnal rotation of the heavens it becomes necessary to relate its instantaneous position to other circles more directly connected with the observer. This is the aim of the remaining three problems which are all stated in a characteristic way underlining the importance of the concepts of the *medium coeli*, and the *ascendant* or horoscope (see page 114). In each case the problem is solved by the same mathematical technique, the angle between the ecliptic and the circle in question being found as an arc on an auxiliary great circle having its pole at the vertex of the angle.

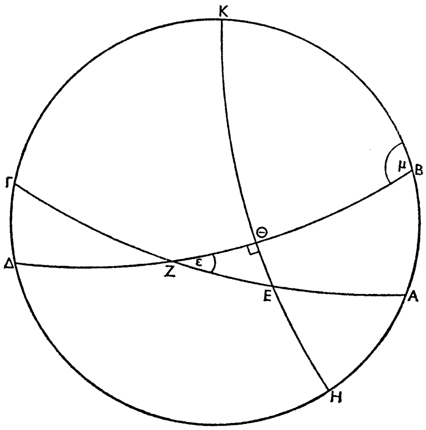


Fig. 4.10

Problem 9: To find the angle between the ecliptic and the meridian.

In the first case [II, 10; Hei 1, 145] we have to find the angle $\mu = \Gamma BZ$ between the ecliptic ΔZB and the meridian $AB\Gamma\Delta$, cf. Figure 4.10 which shows the heavenly sphere seen from the West. The equator is ΓZA and Z is the vernal equinox (Ptolemy uses the autumnal equinox). The position of the ecliptic is given by the longitude $\lambda_e = \lambda(B) = ZB$ of the *medium coeli* B.

With B as centre we now draw an auxiliary great circle KH, intersecting the ecliptic in Θ and the equator in E. Then the arc $K\Theta$ measures the angle to be found. It is obvious that $BH = BE = B\Theta = BK = 90°$, but we also have $KE = 90°$ since

E is the point of intersection of two great circles whose poles both lie on the great circle KBH containing K.

We now consider the spherical triangle ΘZE in which $\Theta = 90°$, $Z = \varepsilon$, and $Z\Theta = \lambda_e - 90°$. By (3.41) we find

$$\tan \Theta E = \sin Z\Theta \cdot \tan \varepsilon$$

or

$$\tan \Theta E = \cos \lambda_e \cdot \tan \varepsilon \tag{4.31}$$

The angle between the ecliptic and the meridian is then determined by the arc

$$\Theta K = 90° - \Theta E$$

which combined with (4.31) gives

$$\cot \mu(\lambda_e) = \cos \lambda_e \cdot \tan \varepsilon \tag{4.32}$$

Thus the angle between the ecliptic and the meridian depends only on the longitude λ_e of the mid-heaven.

Problem 10: To find the angle between the ecliptic and the horizon.

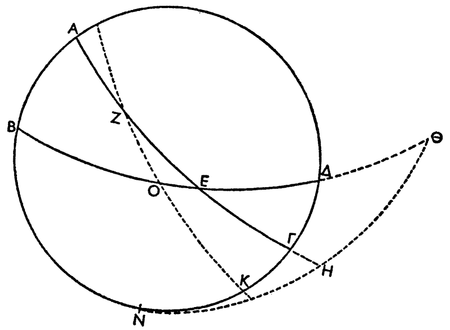


Fig. 4.11

This problem is solved by Ptolemy in a way which gives a good impression of how computations can be carried out by means of the tables of declination and right and oblique ascension. In Figure 4.11 the sphere is seen from the East [II, 11; Hei 1, 154]. The circle ΑΒΓΔ represents the meridian, ΒΕΔ the horizon, and ΑΕΓ the ecliptic. The position of the latter is here given by the longitude $\lambda_h = \lambda(E) = ZE$ of the ascendant, or rising point E, reckoned from the vernal equinox Z. This point is not marked on Ptolemy's drawing, which also omits the equator. On our figure the latter is represented by the dotted circle passing through Z and the East point O of the horizon. It crosses the meridian at K. If the latitude of the observer is φ we have $NK = \varphi$ where N is the nadir point. We have to find the angle $\Delta E \Gamma = \nu$.

Again, we draw a great circle with radius 90° and centre E. It will pass through N

and cut the ecliptic in H and the horizon in Θ, both of which lie on the invisible side of the heavenly sphere if the position of the ecliptic is the one shown here. We then have $EN = EH = E\Theta = 90°$, and furthermore $\Theta N = 90°$, since Θ lies on a circle with N as pole. The angle $\Delta E\Gamma$ is measured by the arc $H\Theta$ which we must try to find. This can be done in several steps. Ptolemy first applies Menelaus' theorem to the two pairs of great circles intersecting at E and N respectively. This amounts to the same thing as applying the formula (3.42) to the spherical triangle $N\Gamma H$ which has a right angle at H. Then

$$\cos N\Gamma = \cos \Gamma H \cdot \cos NH$$

With $NH = 90° - H\Theta$ we find

$$\sin \nu = \sin H\Theta = \frac{\cos N\Gamma}{\cos \Gamma H} \tag{4.33}$$

for the sine of the desired angle ΘEH. The problem is solved as soon as we have determined the arcs $N\Gamma$ and ΓH. And now the method becomes rather complicated.

Let us first consider ΓH. The figure shows that the point O on the equator is rising at the same time as the point E on the ecliptic so that we can determine the arc ZO by the method developed in Problem 8. In other words, we can use the table of oblique ascensions corresponding to the latitude φ and the longitude $\lambda_h = ZE$ to find $ZO = \vartheta(\lambda_h, \varphi)$. We can then find

$$ZK = ZO + 90° = \vartheta(\lambda_h, \varphi) + 90° = \alpha(\Gamma)$$

as the right ascension of the point Γ on the ecliptic which is in lower culmination. Using this value as argument in the table of right ascensions (i.e. the table of oblique ascensions for $\varphi = 0°$) we can then solve the equation

$$\vartheta(\lambda_h, \varphi) + 90° = \vartheta(\lambda(\Gamma), 0°)$$

with respect to the longitude $\lambda(\Gamma)$ of the point Γ. The result is a function of λ_h, connected with the longitude of the mid-heaven by

$$\lambda(\Gamma) = \lambda_e + 180° \tag{4.34}$$

We can also calculate

$$E\Gamma = Z\Gamma - ZE = \lambda(\Gamma) - \lambda_h$$

and

$$\Gamma H = 90° - E\Gamma = 90° - \lambda(\Gamma) + \lambda_h$$

The only remaining quantity is the arc

$$N\Gamma = NK + K\Gamma$$

where $NK = \varphi$ and $K\Gamma = \delta(\Gamma)$. The latter can be found by the table of the obliquity of the ecliptic as the declination corresponding to the longitude $\lambda(\Gamma)$, cf. (4.2) and

the procedure explained in Problem 1. We have accordingly $N\Gamma = \varphi + \delta(\Gamma)$, so that (4.33) can be written

$$\sin v(\lambda_h, \varphi) = \frac{\cos(\varphi + \delta(\Gamma))}{\sin(\lambda(\Gamma) - \lambda_h)} \tag{4.35}$$

Using (4.2) and (4.34) we can introduce the longitude of the mid-heaven and write (4.35) in the form

$$\sin v(\lambda_h, \varphi) = \frac{\cos(\varphi - \delta(\lambda_e))}{\sin(\lambda_h - \lambda_e)} \tag{4.35 a}$$

Although the method is intricate the result is important, particularly in the theory of visibility of the planets (see page 389). The similar problem of determining the first visibility of the New Moon played a very important role in Babylonian astronomy. It is not dealt with in the Almagest where the Egyptian calendar is used and the first appearance of the Moon is without calendrical importance.

Problem 11: To find the position of the ecliptic relative to a vertical.

This is the last of the spherical problems dealt with in the Almagest [II, 12; Hei 1, 160]. It is stated in a characteristic way which once more underlines the fundamental importance of the ecliptic as Ptolemy's preferred circle of reference on the heavenly sphere. In fact, Ptolemy does not begin by defining a given vertical (a great circle through the zenith and nadir points perpendicular to the horizon) by its azimuth in order to proceed with finding the position of the ecliptic relative to such a circle at a given time. Instead he defines upon the ecliptic a point H by its longitude $\lambda = \lambda(H)$. This point will take part in the daily rotation of the heavens and will in each successive position define a vertical. The problem is to find the position of the ecliptic rela-

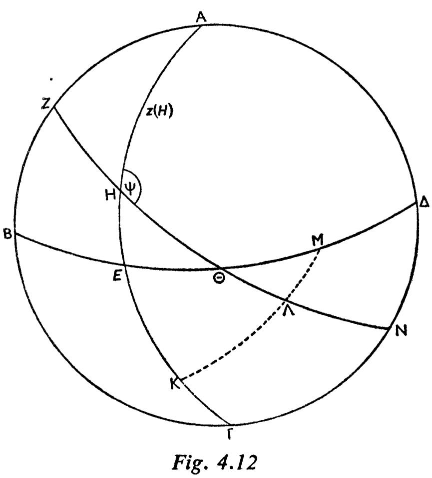


Fig. 4.12

tive to this vertical at a given time t measured in equinoctial hours before or after the upper culmination of H. This implies that we determine both the angle ψ between the ecliptic and the vertical through H, and the position of H itself upon the vertical, expressed by its zenith distance z(H). A special case of the first part of this problem was solved in Problem 9 where we found the angle between the ecliptic and the meridian. The procedure is explained by an example in which Ptolemy determines ψ and z(H) where H is the beginning of Cancer and the time is one hour before culmination [II, 12; Hei 1, 168]. It follows that $\lambda(H) = 90°$, while the observer is placed at Rhodes where $\varphi = 36°$.

In Figure 4.12 we see the meridian ABΓΔ, the eastern part BEΔ of the horizon, and the ecliptic NΘHZ with its culminating point Z (the mid-heaven) and the given point H with the distance $\lambda(H)$ from the vernal equinox, the latter not being shown in the figure.

The first step is to calculate the longitude λ_e and λ_h of the mid-heaven Z and the ascendant Θ by the methods described above (page 114). Given the longitude λ of H we then have the arcs

$$ZH = \lambda - \lambda_e \quad \text{and} \quad H\Theta = \lambda_h - \lambda$$

Next we find the declination $\delta(Z)$ of Z by the table of the obliquity of the ecliptic, which we enter with the longitude $\lambda(Z) = \lambda_e$. We can then find the altitude h(Z) of the medium coeli by

$$h(Z) = BZ = 90° - (\varphi - \delta(Z)) \tag{4.36}$$

This enables us to find the altitude $h(H) = EH$ of H. Here Ptolemy uses the theorem of Menelaos, but we can use a more direct method, applying (3.40) to the spherical triangle ΘEH which has a right angle at E. This gives

$$\sin EH = \sin \Theta H \cdot \sin \Theta$$

In a similar way the triangle ΘBZ gives

$$\sin BZ = \sin \Theta Z \cdot \sin \Theta$$

Dividing these equations we get

$$\frac{\sin EH}{\sin BZ} = \frac{\sin \Theta H}{\sin \Theta Z}$$

or

$$\sin h(H) = \sin h(Z) \cdot \frac{\sin \Theta H}{\sin \Theta Z}$$

For the zenith distance $z(H) = 90° - h(H)$ we have accordingly

$$\cos z(H) = \cos z(\lambda, t, \varphi) = \cos(\varphi - \delta(Z)) \cdot \frac{\sin(\lambda_h - \lambda)}{\sin(\lambda_h - \lambda_e)} \tag{4.37}$$

The determination of the angle AHΘ $= \psi$ is very much as in the previous problem. Ptolemy draws a great circle with centre H and radius 90°, intersecting the vertical in K, the ecliptic in Λ, and the horizon in M. The angle EHΘ $= 180° - \psi$ is measured by the arc KΛ which is calculated by Menelaus' theorem, but more easily found by (3.43) applied to the triangle HEΘ which has a right angle at E. This gives

$$\tan EH = \tan H\Theta \cdot \cos (180° - \psi)$$

or

$$\cos \psi(\lambda, t, \varphi) = -\tan h(H) \cdot \cot (\lambda_h - \lambda) \tag{4.38}$$

or again

$$\cos \psi(\lambda, t, \varphi) = -\cot z(H) \cdot \cot(\lambda_h - \lambda) \tag{4.38 a}$$

where z(H) is determined by (4.37).

It will be very inconvenient to try to eliminate the dependent variables from (4.37) as well as from (4.38), but it can be seen that both the zenith distance and the angle with the vertical through H will be functions of three independent variables t, $\lambda(H)$, and φ. All of these are 'strong' variables for which reason the two functions $z(\lambda, t, \varphi)$ and $\psi(\lambda, t, \varphi)$ can be represented only by tables with three separate entries. These tables are constructed in the following way:

First Ptolemy selects seven latitudes φ_n ($n = 5, 7, \ldots 17$), one corresponding to every other parallel (page 108) from Meroë ($\varphi_5 = 16°;27$) to the mouth of Borysthenes ($\varphi_{17} = 48°;32$), outside which zone he obviously seems to doubt the existence of astronomers needing such tables. Then he selects twelve λ-values $\lambda_m = (m - 1) \cdot 30°$ corresponding to the beginnings of each of the twelve signs on the ecliptic. Finally the variable t is represented by a set of values t_i beginning at noon, and continuing with each whole hour before and after noon, the last values being, for instance, $t_7 = 6^h$ and $t_8 = 6^h42^m$ if the point of the ecliptic under consideration has a half diurnal arc of 6^h42^m under the latitude in question. In other words, Ptolemy considers only positions of H above the horizon.

The values $z(\lambda_m, t_i, \varphi_n)$ and $\psi(\lambda_m, t_i, \varphi_n)$ are then calculated and sorted out so that there is a particular table of constant latitude for each of the seven values of φ_n. Inside each of these tables there are 12 subtables of constant longitude λ_m, arranged according to the order of the signs, and beginning with Cancer as the sign into which the Sun is entering on the longest day of the year. Each of the 84 subtables has 4 columns: the first contains t_i, the second $z(\lambda_m, t_i, \varphi_n)$, and the third and fourth $\psi(\lambda_m, t_i, \varphi_n)$ for points H which at the time t_i lie East and West of the meridian respectively. The complete set of tables fills one separate chapter of the Almagest [II, 13; Hei 1, 174].

The calculation of the function values, and their arrangement in the various tables, are made easier by certain symmetrical properties of z and ψ which Ptolemy proves by geometrical methods in the introduction to this problem [II; 12, Hei 1, 160]. They

can be stated as follows: If H_1 and H_2 are two symmetrical positions of H relative to the meridian corresponding to the same interval of time before and after noon, we have

$$z(H_1) = z(H_2)$$

and

$$\psi(H_1) + \psi(H_2) = 180°$$

As we are going to see (page 219), the main use of these tables is for the computation of the diurnal parallax of the Moon and the Sun [cf. V, 19; Hei 1, 444].

CHAPTER 5

The Motion of the Sun

Introduction

The mathematics and the spherical astronomy of the first two Books of the Almagest contained what Ptolemy considered the preliminaries necessary to the study of planetary theory, which is the main subject of the whole work, and that part of astronomy in which Ptolemy asserts his own originality in the most convincing way.

Here Apollonius and Hipparchus had laid the foundation of the first workable geometrical models using eccentric circles and epicycles (see p. 134 f.). But nobody can deny that Ptolemy, in this field, proved to be not only a worthy successor of these great theoreticians of the past, but a highly competent, skilled and ingenious theoretical astronomer himself. In the art of devising geometrical theories of planetary motion and adapting them to observations in such a way that a considerable number of celestial phenomena could be deduced, Ptolemy was unique. In fact, his skill was not surpassed until Kepler once and for all broke the spell of uniform circular motion and discarded the main kinematical tool upon which Ptolemaic astronomy was founded.

Ptolemy himself was always careful to acknowledge the work of his predecessors in the most conscientious way (Toomer 1970, p. 190). Thus it is from the Almagest that we know many of Hipparchus' and almost all of Apollonius' contributions to theoretical astronomy. One even has the impression that Ptolemy regarded it as his own vocation simply to continue and to complete their work, solving problems which they had left unsolved, and developing theories of phenomena of which they had been unable to give a mathematical account. Thus the theory of the motion of the Sun in Book III, and the first attempt to give a theory of the Moon in Book IV, are mainly the work of Hipparchus. But from Book V Ptolemy takes the lead with the later theories of the remaining planets. Thus the bulk of the Almagest is due to his own industry and scientific ingenuity, as the following chapters will reveal.

The Order of the Planetary Theories

Apart from Books VII and VIII (on the fixed stars), all the rest of the Almagest is devoted to planetary theory. Here the most obvious feature is that each planet is dealt with separately. There is no general theory of the solar system as such, although the superior planets Mars, Jupiter, and Saturn are treated according to the same general

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_5,

scheme, their models differing in numerical parameters only. The theory of Venus is slightly different, and those of the Moon and Mercury are greatly so; therefore, in a strict sense, it is without meaning to speak of the Ptolemaic theory of the planets as such.

The order in which Ptolemy deals with the various theories does not follow the accepted order of the planets, i.e. the Moon, Venus, Mercury, the Sun, Mars, Jupiter, and Saturn reckoned upwards from the Earth (see p. 295). Instead he begins with the theory of the Sun, which occupies the whole of Book III. The reason is that the motion of the Sun appears to be essential to all the other planets including the Moon. To be more precise, the solar year enters as a characteristic period into the component movements of all the planets and becomes the only numerical parameter common to all the models.

This is a simple fact the explanation of which is outside the scope of the Almagest. It is easy to understand if we regard the solar system from a heliocentric (Copernican) point of view. Since the planets are observed from the Earth, their apparent positions will depend upon two factors, viz. where the planet itself is, and where the Earth is. The first is determined by the motion of the planet in its own orbit, but the second upon the motion of the Earth in its path around the Sun during the period of one year. The latter motion is reflected as the motion of the Sun around the Earth in exactly the same period if we look upon the system from a geocentric point of view. Thus we must in any case have a certain correlation between the motion of the Sun and the motion of any of the planets. There are therefore excellent astronomical reasons for starting with the theory of the Sun. They also agree well with the prevailing astrological belief in the excellence of the Sun as the one among the heavenly bodies which has the strongest influence upon the Earth.

This solar theory is developed in Book III, in which Ptolemy without hesitation acknowledges his debt to Hipparchus – *that indefatigable and truth-loving scientist Hipparchus* [III, 1; Hei 1, 191] – to whom all the main features of the solar theory seem to be due. The theory has the form of a geometrical model, the parameters of which are deduced from observations in such a way that the principal phenomena of the motion of the Sun can be reproduced, first and foremost its varying velocity throughout the year. A number of these observations were made by Ptolemy himself, but many more were taken from Hipparchus 300 years before, and presumably found in the collections of the Alexandrian library which contained also many much older observations from Babylonian and other sources. In the following we shall give a brief outline of how the model is defined, how its parameters can be found from the observations, and how practical computations and predictions can be made from the model by means of tables developed for this purpose[1]).

1) There are many earlier expositions of the theory of the motion of the Sun among which we may mention the writings of Delambre (II, pp. 99 ff.), Herz (1887,I), Tannery (1893, pp. 142–178), Dreyer (1906, pp. 148 ff.), Rome (1937–38 and 1943), Stumpff (I, 15 ff.) and Petersen and Schmidt (1967).

Time-Reckoning in the Almagest

As soon as we begin to consider the fundamental observations used by Ptolemy, we are presented with the intricate problem of chronology. The time of a particular observation is given in terms which to-day seem strange and hard to understand. Thus Ptolemy states that *in the year 463 after the death of Alexander the summer solstice was on the 11th Mesore about 2 hours after midnight on the 12th* [III, 1; Hei 1, 206]. Another example is the statement that the autumnal equinox occurred *in the third year of Antoninus* ⟨Pius⟩ *on the 9th Athyr about one hour after Sunrise* [ibid. Hei 1, 204]. In both cases the time in question is defined by four elements

1 the hour of the day
2 the date, or the double date
3 the month
4 the year

The Almagest itself contains no particular treatise on time-reckoning. But in order to be able to interpret the many observations, we must now briefly examine Ptolemy's method of dating as it can be inferred from the many instances of dated observations quoted in the text.

Here it is essential to remember that Ptolemy lived and worked in Egypt. Furthermore, in spite of all that has been said of the influence of Greek culture upon the schools of Alexandria, and upon Hellenistic culture in general, it clearly appears from many places in the Almagest, and other sources as well, that Greek chronology and time reckoning had never succeeded in replacing the traditional Egyptian calendar. The latter survived with almost no traces of Greek influence, presumably because of its simplicity and superiority over the complicated Greek systems. Therefore it is the Egyptian calendar which is the chronological basis for the principal work of Hellenistic astronomy (cf. Ginzel I, p. 150 ff., and Parker 1950).

The Hour of the Day

Among the four elements of a point of time mentioned above let us first consider the hour of the day. The division of the natural day into 24 hours still in use seems to be a legacy from Egyptian astronomy. But this division could be made in two different ways, which may be termed the civil and the astronomical, respectively. According to the civil system the natural day was first divided into the two main periods of day and night. Each of these were then subdivided into 12 equal periods called hours of the day, and hours of the night. These are the seasonal hours which vary in length throughout the year, except for observers placed on the terrestrial equator (see above page 114). The astronomical system is simpler and essentially the same as we use to-day. Here the whole of the natural day is divided into 24 equal, or equinoctial, hours. It follows that the latter can be used to measure the varying lengths of the

seasonal hours by a method which we already know from the chapter on spherical astronomy, cf. (4.27–28). If nothing else is said it can be assumed that the hours used in the Almagest are the astronomical, equal, or 'equinoctial' hours. This is the case with all astronomical data, and we shall not refer to seasonal hours at all in the following chapters.

The next question is how the hours were numbered, or – which amounts to the same thing – when the natural day began according to the Egyptians. It is well known that the Egyptian civil calendar reckoned the new day from sunrise, or perhaps from dawn when the light began to return. All statements in the Almagest are compatible with the hypothesis that Ptolemy always followed the usage of the civil calendar, thus reckoning the new day from dawn and numbering the hours accordingly[2]). This is a fact, notwithstanding the other fact that Ptolemy computed all his astronomical tables for a standard epoch defined as *noon on the first day of the Egyptian month Thoth in the first year of the reign of Nabonassar* [III, 7; Hei 1, 256]. This has often been expressed in the phrase that here he reckoned the astronomical day as starting from noon. This had led to some confusion, and if one wishes to talk of an astronomical day in Ptolemy at all, it should be noticed that such a day never interferes with the numbering of dates, the latter being always reckoned according to the traditional Egyptian usage.

The Day of the Month

Concerning the problem of dates, i.e. of numbering the days of the month, we must remember that the Egyptian calendar supposed the year to be 365 days. These were divided into 12 equal months of 30 days each, named

1 Thoth
2 Phaophi
3 Athyr
4 Choiak
5 Tybi
6 Mechir
7 Phamenoth
8 Pharmuti
9 Pachon
10 Payni
11 Epiphi
12 Mesore

This accounts for 360 days. The remaining 5 were regarded as surplus or epagomenal days and placed at the end of the year (Ginzel I, p. 171 f.). This is the system adopted in the Almagest, where the only problem concerning dates is the curious use of

2) This has been proved by Petersen and Schmidt (1967); see in particular the section *On Ptolemy's Use of the Egyptian Calendar*, pp. 91 ff.

double dates in a number of cases, of which one was quoted above (page 124). The expression *on the 11th Mesore after midnight on the 12th*, and similar double dates, have confused some authors. Thus Manitius [I, 431, note 27] seems to think that this usage has something to do with the discrepancy between a civil day beginning at dawn, and a supposed astronomical day beginning at noon. That this is a misunderstanding has been proved by O. Neugebauer and H. B. van Hoesen (1959, p. 167). Actually Ptolemy makes use of double dates mostly when speaking of hours after midnight – for reasons not known to us – and the statement quoted above should rather be translated as *on the 11th Mesore after midnight, towards the beginning of the 12th*. This makes everything clear, and in the following we shall assume that a double date refers always to the first of the two dates mentioned and is understood as a date in the civil Egyptian calendar.

The Year

We are then left with the problem of the year. The civil Egyptian year of 365^d was too short compared with the tropical year of about 365^d6^h. This meant that year after year the civil New Year started about 6^h too early, with the obvious consequence that no correlation could be established between the civil year and the seasons of the tropical year. Actually, New Year's day travelled through all the seasons, performing a complete revolution in 1461 Egyptian years – the so-called Sothis period (Ginzel I, p. 181 f.). After the expiration of such a period the seasons will again begin on the same dates as before. In 238 B.C. the priests at Canopus in the Nile delta (cf. page 12) tried to reform this calendar, giving 366^d to every fourth year (Ginzel I, p. 196 f.), but this came to nothing, and it was not until 46/45 B.C. that a calendar with such a leap year was introduced by Julius Caesar, with dates reckoned from midnight according to the Roman usage. But although this Julian calendar was introduced some 200 years before the time of Ptolemy, it left almost no trace in the Almagest, where a year, in general, meant one Egyptian year of 365^d.

The Epochs

There is no single answer to the question how the years were numbered in the Almagest. The ancient method used in Egypt, Israel, and Mesopotamia was to consider the reign of a king as a definite period inside which one numbered the years elapsed since his accession to the throne. This method presupposes that one knows the exact reigns of the successive kings. In his *Canon Basileion* Ptolemy has given us a table of the reigns of Babylonian, Persian, Macedonian, and Egyptian kings, and Roman emperors, beginning with Nabonassar and ending with Antoninus Pius. It is reproduced in Ginzel I, 139.

This official method of reckoning years is very inconvenient and confusing when one has to calculate long intervals of time, as is often the case in astronomy. The Almagest reveals that Ptolemy tried to reduce the confusion by selecting a few standard epochs.

A single epoch would have been more convenient, but would have prevented Ptolemy from quoting earlier observations literally in the form in which they were handed down from other astronomers. The most prominent of the eras used in the Almagest are the following [cf. III, 7; Hei 1, 256]:

1) The era of Nabonassar, which began at noon on the first day of the month Thoth in the first year of the reign of Nabonassar[3]). To this epoch all Ptolemy's tables of mean motions of the planets are referred (see e.g. page 290), perhaps because he maintains that from this time onwards almost all observations are said to have been handed down to his own time [III, 7; Hei 1, 254]. This is in accordance with recent knowledge of Babylonian astronomy (see v. d. Waerden 1965, p. 92).

2) The era of Philippus which began at noon Thoth 1 in the first year of Philippus Aridaeus. It is also the era reckoned from the death of Alexander the Great. The interval between the two epochs is 424 Egyptian years.

3) The era of Augustus, which began at noon Thoth 1, in the first year of Augustus, 718 Egyptian years after Nabonassar. This epoch is used only once in the Almagest.

4) Various eras reckoned from the first year of various Roman emperors of Ptolemy's own time or immediately before, such as Hadrian and Antoninus Pius.

The following table gives a summary of this material. The intervals between the various epochs are given in Egyptian years as quoted in the Almagest. The dates to the left are the date of the first day of the month Thoth of the year in question, expressed in the Julian calendar, as computed by Ginzel. Formulas for the reduction of dates from one era to another have been published in several manuals of chronology. A simplified version is due to B. L. van der Waerden (1956).

Date	Era of	Egyptian year of Nabonassar	Philippus	Augustus
B.C. 747, Febr. 26	Nabonassar			
B.C. 324, Nov. 12	Philippus	424		
B.C. 30, Aug. 31	Augustus	718	294	
A.D. 116, July 25	Hadrian	863	439	145
A.D. 137, July 20	Antoninus	884	460	166

Of a completely different kind and origin than these arbitrary eras is another system of numbering years occasionally referred to in the Almagest [III, 1; Hei 1, 195 ff.]. It is founded upon the so-called Calippic cycle, named after the 4th century Athenian astronomer Calippus, who is known also from his revision of the planetary theory of Eudoxus (Heath 1913, p. 295 f., cf. Duhem I, 123). The Calippic cycle arose from a problem unknown to the Egyptians who, in their calendar, cared nothing about the Sun or the Moon. On the other hand, the Greeks had a lunisolar calendar

3) Nabonassar reigned 747–735 B.C. His name means Nebu Protector, Nebu being the Assyrian name of the old Babylonian god of writing and learning. For this reason the name was common in Mesopotamia, and temples were dedicated to Nebu in astronomical centres like Babylon and Borsippa.

based partly upon the 'tropical' year (believed to be 365^d6^h) and partly upon the mean synodic month. Since the latter is $29^d.53$, an actual Greek month was either full (containing 30^d) or hollow (containing 29^d). The problem was to devise a calendar creating so much agreement between these two periods that the seasons did not revolve through the year as in Egypt, but as far as possible were tied to the same month every year.

Already in the 5th century this problem was attacked by another Athenian astronomer by the name of Meton, who observed in Athens in 432 B.C. (Heath 1913, p. 293). His idea was to base a calendar upon the relation

$$6940 \text{ days} = 19 \text{ tropical years} = 235 \text{ mean synodic months}$$

so that there were 125 full and 110 hollow months in the period. These were distributed inside this Metonic cycle according to a fixed rule so that 12 years had 12 months each while 7 years had 13 months. The result was that months and seasons kept pace with sufficient exactness. The relation defining the Metonic cycle implies that the tropical year is

$$6940^d : 19 = 365 + \tfrac{1}{4} + \tfrac{1}{76} \text{ days}$$

i.e. a little more than the accepted value, so that 4 Metonic cycles, or $4 \cdot 6940^d = 27760^d$ were supposed to contain one day too much. What Calippus proposed, therefore, was to cut one day, putting

$$76 \text{ years} = 27759 \text{ days}$$

This would give a tropical year of exactly $365\frac{1}{4}$ days (Ginzel II, p. 409 f.). We do not know at what place inside this 76 year Calippic cycle the superfluous day was discarded – it was presumably at the end of the period. Likewise the exact distribution of long and short years inside the cycle is a little uncertain (Ginzel II, p. 413 and 415 f.). But we know that each cycle began at midsummer time, like the Greek New Year, i.e. sometime between June 19 and July 18. In the Almagest [VII, 3; Hei 2, 32] it is stated that a certain date in the 36th year of the 1st Calippic period coincided with a date in the year Nabonassar 454. Thus this first Calippic cycle must have begun in the year of Nabonassar 454 − 36 = 418, i.e. in the year which began in 330 B.C. This method of time-reckoning was never used much for civil purposes in Greece but seems to have been adopted by astronomers. Thus Hipparchus' observations of autumnal and vernal equinoxes refer to the 3rd Calippic period, at least in the form in which they have survived in the Almagest, as we shall see in the following sections [cf. III, 1; Hei 1, 195].

The Tropical Year

After this digression on time-reckoning we now return to the solar theory as it is developed in the Almagest. It begins with a lengthy discussion of the fundamental

constant of any theory of the motion of the Sun, viz. the year [III, 1; Hei 1, 191–209][4]). There are here three main questions:

Which kind of year is essential to the theory?
Has this year a constant length?
What is the length of the year?

With respect to the first question, Ptolemy is, of course, aware that the Egyptian year of 365^d is devoid of any astronomical significance. This appears from the fact that it leads to a calendar in which the months are not correlated to the physical effects of the Sun's motion, i.e. to the changing seasons. Since one of the main purposes of a solar theory must be to account for the seasons as produced by the annual motion of the Sun, the Egyptian year can have no place in the theory.

The empirical fact underlying the whole of this matter is that the four seasons are connected with the Sun's apparent motion along the ecliptic. If this motion is described by the ecliptic longitude $\lambda_{\odot}(t)$ of the solar centre, reckoned from the vernal equinox ♈, we have the beginning of

Spring when $\lambda_{\odot}(t) = 0°$ (Vernal equinox)
Summer when $\lambda_{\odot}(t) = 90°$ (Summer solstice)
Autumn when $\lambda_{\odot}(t) = 180°$ (Autumnal equinox)
Winter when $\lambda_{\odot}(t) = 270°$ (Winter solstice)

The seasons defined in this way are the astronomical seasons. It is a matter of experience that they correspond fairly well to the climatic seasons on the northern hemisphere. From the astronomical point of view the essential feature is that the Sun moves 90° along the ecliptic during each season.

We can now define one tropical year as the period of time in which the Sun returns to the same equinox. Ptolemy argues that it is the tropical year which is essential to the solar theory, since this year alone will be able to uphold the correspondence between the changing seasons and the time of the year.

Like Hipparchus, Ptolemy knows also what we now call the sidereal year, or the period of return to the same fixed star [III, 1; Hei 1, 192]. But whereas the tropical year is shorter than $365\frac{1}{4}$ days (see below) the sidereal year is found to be longer. This is an effect of the precession of the equinoxes (to adopt the modern phrase), that is of the slow motion of the equinoctial points from East to West upon the ecliptic, due to the gyroscopic spinning of the Earth around its own axis. Ptolemy has a different explanation, ascribed to Hipparchus, to whom it is the sphere of the fixed stars – here Ptolemy is speaking of spheres – which has a proper motion towards the East, relative to a fixed ecliptic and its equinoxes. This is the starting point of the Ptolemaic theory of precession, which is developed in more detail later on [VII, 2–3; Hei 2, 12 ff., cf. page 239]. Here Ptolemy mentions it only in passing, in order to

4) For this section see Tannery (1893, pp. 142–160) and Rome (1937 and 1938).

Solar Observations in the Almagest

	Observation	Date According to to Ptolemy	Julian Date According to Manitius	Ref. No.
S_1........	Summer Solstice	Apseudes Phamenoth 21 Morning	− 431 June 27	8
S_2........	Summer Solstice	Calippos I 50 End of the year	− 279	16
S_3........	Autumnal Equinox	Calippos III 17 Mesore 30 Sunset	− 161 Sept 27	34
S_4........	Autumnal Equinox	Calippos III 20 Epag. 1 Morning	− 158 Sept 27	35
S_5........	Autumnal Equinox	Calippos III 21 Epag. 1 Noon	− 157 Sept 27	36
S_6........	Autumnal Equinox	Calippos III 32 Epag. 3 Midnight	− 146 Sept 26	37
S_7........	Vernal Equinox	Calippos III 32 Mechir 27 Morning	− 145 March 24	38
S_8........	Autumnal Equinox	Calippos III 33 Epag. 4 Morning	− 145 Sept 27	40
S_9........	Autumnal Equinox	Calippos III 36 Epag. 4 Evening	− 142 Sept 26	41
S_{10}.......	Vernal Equinox	Calippos III 43 Mechir 29 Midnight	− 134 March 23	45
S_{11}.......	Vernal Equinox	Calippos III 50 Phamenoth 1 Sunset	− 127 March 23	47
S_{12}.......	Autumnal Equinox	Hadrian 17 Athyr 7 2^h after noon	+ 132 Sept 25	62
S_{13}.......	Autumnal Equinox	Antoninus 3 Athyr 9 1^h after sunrise	+ 139 Sept 26	89
S_{14}.......	Vernal Equinox	Antoninus 3 Pachon 7 1^h after noon	+ 140 March 22	91
S_{15}.......	Summer Solstice	Antoninus 3 Mesore 11 2^h after midnight	+ 140 June 25	92

explain why the sidereal year is unable to keep pace with the seasons, so that it must be discarded as a fundamental astronomical notion in the theory of the Sun[5]).

The Length of the Year

The next question is whether the tropical year is an astronomical constant. This can be decided only by observations over a long period of time. From the general table of observations (Appendix A) we select the following 15 numbers[6]), upon which the whole solar theory of the Almagest is founded (cf. the lists in Rome 1937, p. 218, and Petersen and Schmidt 1967, p. 84).

Of these observations S_3–S_{11} are due to Hipparchus, who deduced from them that the length of the tropical year is less than the $365\frac{1}{4}$ days commonly assumed by the mathematicians. How Hipparchus reduced the observations is not quite clear from the Almagest; but it seems that he used also the older observation S_2 made by Aristarchus in 280 B.C. He dealt with the problem in two (now lost) books, *On the Length of the Year*, and *On Intercalary Months and Days*, from which Ptolemy quotes the final result [III, 1; Hei 1, 207]

$$\begin{aligned} 1 \text{ tropical year} &= 365 + \tfrac{1}{4} - \tfrac{1}{300} \text{ days} \\ &= 365^d;14,48 \\ &= 365^d5^h55^m12^s \end{aligned}$$

Compared with the modern value $365^d5^h48^m46^s$ this Hipparchian year is only about 6 minutes too long. The error is due partly to imperfect instruments, partly to the influence of atmospheric refraction.

The four last observations S_{12}–S_{15} were made by Ptolemy[7]) himself [III, 1; Hei 1, 194 f.] in order to see whether Hipparchus' value was still correct after a lapse of almost 300 years. This was done in the following way:

5) Because Ptolemy assumed a constant rate of precession (see page 248) his tropical year became an astronomical constant and there was no inconvenience in referring stellar positions to the vernal equinox. Later Arabic astronomers called the Ptolemaic theory of precession in question and advocated the sidereal or the anomalistic year as the fundamental, astronomical unit of time. An early example of this criticism is the treatise *De motu Solis* by the 9th century Baghdad astronomer Thābit ben Qūrra (ed. Carmody, 1960, pp. 63–79); a late instance is Copernicus' use of the fixed stars as a frame of reference instead of the vernal equinox (*De Rev.*, II, 14), and his discussion of the sidereal year (*De Rev.*, III, 13).
6) The Julian dates of these observations are taken from Manitius' translation of the Almagest. More recent calculations have resulted in a number of corrections to which references are given in Appendix A.
7) The only instrument for this purpose described by Ptolemy [III, 1; Hei 1, 195] is a ring of bronze with a diameter of 2 cubits and placed in the plane of the equator. At the time of an equinox the shadow of one half of the ring falls exactly upon the other half (cf. Dicks 1954, p. 79). The instrument was still used by Theon of Alexandria (see Rome 1926, p. 11). – Theon (ed. Rome, p. 817) assumed that both Hipparchus and Ptolemy had found equinoxes and solstices by the meridian instrument mentioned by Ptolemy [I, 12; Hei 1, 64] in connection with the obliquity of the ecliptic and described in great detail by Proclus (*Hyp.* III, 1). This view is supported by Rome (1937, p. 218 f.) but it seems impossible that solstices were determined in this way because of the slow variation af the declination of the Sun just before the maximum (or minimum). Also the fact that the time of the summer solstice S_{15} is given as two hours after midnight reveals that it was the result of some kind of calculation, based perhaps on measurements of corresponding altitudes.

Ptolemy first compared the observations S_6 and S_{13}. Here we have two autumnal equinoxes separated by a considerable interval of time. The question is, how long is this interval?

If we compute by means of the Callippic calendar, assuming the year to be exactly $365\frac{1}{4}$ days, we find that the two events are separated by 285 Egyptian years containing a surplus of $71\frac{1}{4}$ intercalary days, i.e. a total number of days equal to

$$285 \cdot 365 + 71\tfrac{1}{4}$$

But we can also compute the interval directly from the observations expressed in the Egyptian calendar. This gives a number of days equal to $285 \cdot 365 +$ the interval from midnight Epag. 3 to 1 hour after sunrise Athyr 9. The latter interval is

$$1^d6^h + 30^d + 30^d + 9^d + 1^h = 70^d7^h$$

For this value Ptolemy substitutes $70^d7^h15^m$, which is no great error considering the uncertainty of the observations. This enables him to express this number of days as

$$70 + \tfrac{1}{4} + \tfrac{1}{20}$$

This does not agree with the $71\frac{1}{4}$ days found by means of the Callippic periods. The only explanation is that the Callippic year of $365\frac{1}{4}$ days is wrong, the error amounting to

$$71\tfrac{1}{4} - (70 + \tfrac{1}{4} + \tfrac{1}{20}) = 19/20 \text{ day}$$

in 285 years. This corresponds to one day in 300 years; but this was what Hipparchus had found as the excess of the Callippic year over the tropical year.

In a similar way the observations S_7 and S_{14} are compared, while S_{15} is taken together with the observation S_2 made in 280 B.C. by Aristarchus, and with the still older observation S_1 by Meton and Euctemon in 432 B.C. The result is the same as before, and Ptolemy is therefore satisfied that

1) the tropical year is an astronomical constant, and
2) its length is $365^d;14,48$ as found by Hipparchus

The fact that this value is a little too high (see page 131) has many consequences for the various theories of planetary motion. In particular it tends to make the mean daily motions in longitude too small. For a more detailed account of this first analysis of the motion of the Sun in the Almagest see Rome (1938, p. 6 ff.), where also Theon's interpretation of the method is explained.

The Mean Motion of the Sun

Concerning the motion of the Sun we already know the following empirical facts:

1) The apparent path of the Sun on the heavenly sphere is a great circle called the ecliptic, or the *oblique circle* [I, 8; Hei 1, 28]. It intersects the celestial equator

in the vernal and autumnal equinoctial points [I, 8; Hei 1, 29], forming with the equator an angle $\varepsilon = 23°;51,20$ [I, 12; Hei 1, 68].

2) The Sun has a constant period of return to the vernal equinox equal to 1 tropical year [III, 1; Hei 1, 208] or $365^d;14,48$.

3) Finally it is a fact that the apparent annual motion of the Sun has a varying velocity. That this is so was deduced by earlier observations from the unequal lengths of the four seasons. As mentioned above (p. 129) each of the seasons corresponds to a motion of the apparent Sun through an arc of 90° of the ecliptic. Since the Sun goes through these equal arcs in unequal times, its apparent motion is not uniform but vitiated by an anomaly, or irregularity, i.e. a deviation from uniform motion.

At first Ptolemy takes only the first two of the phenomena above into account, calculating a quantity

$$\omega_{\odot} = \frac{360°}{365;14,48} = 0°;59,8,17,13,12,31 \text{ per day} \tag{5.1}$$

which represents the number of degrees the Sun would move per day if it had a uniform motion along the ecliptic. Accordingly $\omega_{\odot}$ is the mean angular velocity of the Sun. Because of the erroneous value of the length of the tropical year (page 132) $\omega_{\odot}$ is a little too small. Another way of interpreting it is to define an e c l i p t i c m e a n S u n as a point moving with uniform velocity upon the ecliptic and performing one complete revolution in one tropical year. The mean angular velocity of the real Sun is, therefore, equal to the constant angular velocity of the mean Sun.

Denoting the ecliptic longitude of the mean Sun at the time t by $\lambda_m(t)$, we can express its uniform motion by the relation

$$\lambda_m(t) = \lambda_m(t_0) + \omega_{\odot}(t - t_0) \tag{5.2}$$

where t_0 is an arbitrary epoch to be defined later. The term $\omega_{\odot}(t - t_0)$ is calculated by Ptolemy for various values of the interval $(t - t_0)$ and tabulated with $(t - t_0)$ as argument. In this way appears a set of solar tables, or *Tables of the Uniform Motion of the Sun* [III, 2; Hei 1, 210–215], comprising

1) a table with intervals of 1^h from 1^h to 24^h
2) a table with intervals of 1^d from 1^d to 30^d
3) a table with intervals of 30^d from 30^d to 360^d
4) a table with intervals of 1^a from 1^a to 18^a
5) a table with intervals of 18^a from 18^a to 810^a

In each table the values are given to 6 sexagesimal places after the integer number of degrees (in which multiples of 360° are discarded). This exactitude is, of course, exaggerated, and Ptolemy always makes use of rounded off values when comparing observations with theoretical results.

It should be noticed that the so-called *Handy Tables* (see p. 397) contain a set of solar tables very similar to those of the Almagest, the main difference being that the

values are given with only one or two sexagesimal places, and that the table with 18-year intervals in the Almagest is replaced by a table with intervals of 25 Egyptian years (cf. Neugebauer 1951, p. 90). Tannery (1893, p. 165) suspected that the 18-year interval is used as only slightly different from the Saros period of 18^a11^d (see p. 162) while 25 Egyptian years are very nearly equal to 309 synodic months.

General Consideration of Geometrical Models

So far we have constructed a preliminary theory in which the Sun is represented by a mean Sun moving with constant velocity along the ecliptic according to the relation (5.2). But this latter relation is unable to give more than an approximative description of the non-uniform motion of the real Sun. Therefore the next step must be to replace the preliminary theory with a more refined model taking the anomaly of the solar motion into account. Ptolemy was aware that earlier Greek astronomers had developed at least two such models, both of which respected the principle of uniform circular motion in the strict sense.

These earlier models, or hypotheses, are carefully examined in the Almagest [III, 3; Hei 1, 216] which is, in fact, the only remaining source of what we know about them. The following account differs from that of Ptolemy only in so far as we shall use a vector notation. This was, of course, unknown to him, but is a convenient means of expressing his geometrical arguments in a shortened form.

The Eccentric Model

The first, or eccentric model, can be characterized by the relation

$$\overrightarrow{TP} = \overrightarrow{TD} + \overrightarrow{DP} \tag{5.3}$$

between 3 vectors which all lie in the plane of the ecliptic (cf. Figure 5.1). They are defined in the following way.

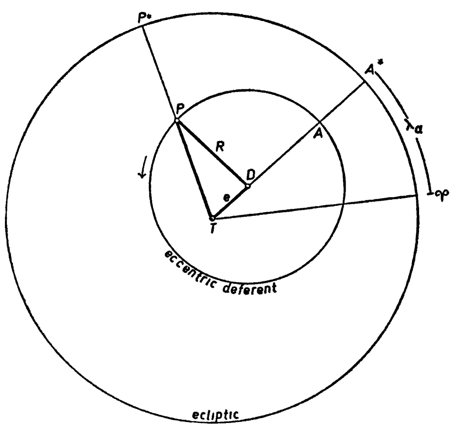


Fig. 5.1.

The vector $\overrightarrow{TD}$ is the eccentricity vector connecting the centre T of the Earth with a fixed point D. It has a constant length e called the eccentricity and a fixed direction in space, pointing to a point A* on the ecliptic with the longitude λ_a reckoned from the vernal equinox ♈.

The vector $\overrightarrow{DP}$ may be called the deferent vector. It connects D with the centre P of the Sun. It has a constant length R, and rotates from West to East with a constant angular velocity equal to the mean angular velocity $\omega_\odot$ of the Sun defined by (5.1), completing a revolution relative to ♈ in one tropical year.

Finally $\overrightarrow{TP}$ is the geocentric position vector of the Sun, connecting the centre of the Earth with that of the Sun. In the following we shall disregard all effects of parallax (see page 214) and suppose the observer to be placed at T, so that TP is the line of sight from the observer to the (centre of) the Sun. TP intersects the ecliptic in a point P* with the longitude $\lambda(t)$.

From the properties of the vector $\overrightarrow{DP}$ follows that the Sun P moves upon a circle with centre D and radius R. This circle 'carries' the Sun, and the Latin astronomers of the Middle Ages called it *circulus deferens solis* (or *solem*). Ptolemy calls it the eccentric circle; in the following we shall refer to it as the deferent[8]). The point A (with the longitude λ_a) is called the apogee because here the Sun has its maximum distance (R + e) from T. The apogee is, as it were, the 'top' of the eccentric circle; Latin astronomers usually called it the *aux*, or the *aux in prima significatione*, the *aux in secunda significatione* being the projection A* of A upon the ecliptic. Opposite to A is the perigee Π (in Latin the *oppositum augis*) where the Sun has its minimum distance (R − e) from T.

Since $\overrightarrow{DP}$ rotates with constant angular velocity about D, it follows that $\overrightarrow{TP}$ will rotate about T in the same direction, but with a varying angular velocity, so that $\lambda(t)$ is a non-linear function of time. Thus a uniform circular motion of P on the deferent has produced a non-uniform motion of P* on the ecliptic. The hypothesis implied in this model can now be stated as follows: It is possible to assign such numerical values to the parameters e, R, and λ_a that P* will mark the apparent position of the Sun on the ecliptic at any given time t.

The eccentric model is usually ascribed to Hipparchus, since Ptolemy relates [III, 4; Hei 1, 233] that Hipparchus found

$$e/R = 1/24 \quad \text{and} \quad \lambda_a = 65°;30$$

However, Ptolemy states only that Hipparchus used the model, not that he invented it. It may well be of an earlier date. Perhaps Apollonius knew it (see page 340).

8) The originally extremely confused Latin terminology of Mediaeval astronomers became more settled during the 13th century. In a recent paper (Pedersen 1973) I have published a 15th century glossary of the most widely used terms, found in a MS in St. John's College, Cambridge. This glossary is derived from the anonymous *Theorica Planetarum* which from about the middle of the 13th century spread a standard terminology in all European universities (see Pedersen 1962).

The Epicycle Model in General

The second model by which a uniform rotation can produce a non-uniform, apparent motion on the ecliptic is characterized by the relation

$$\overrightarrow{TP} = \overrightarrow{TC} + \overrightarrow{CP} \tag{5.4}$$

and illustrated by Figure 5.2.

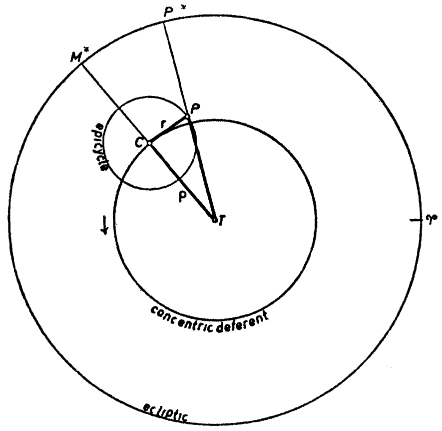


Fig. 5.2.

Here the deferent vector $\overrightarrow{TC}$ has a constant length ρ. It rotates from West to East with a constant angular velocity $\omega_{\odot}$ around the centre T of the Earth. Thus the point C will describe a circle with radius ρ, concentric with the Earth and called the deferent of C.

The vector $\overrightarrow{CP}$ is the epicycle vector. It has a constant length r and performs a uniform rotation about C with the constant angular velocity Ω. Consequently the point P (representing the Sun, or another planet) can be said to describe a circle with centre C and radius r, called the epicycle, in the same way as the epicycle centre C describes the deferent. For the epicycle the Latin astronomers of the Middle Ages used the term *epicyclus*, but sometimes they also called it *circulus brevis* (cf. page 89).

It is seen that the position vector TP of the planet P will rotate about T with a varying length and a non-uniform angular velocity. It points to a point P* on the ecliptic, the longitude λ(t) of which will be a non-linear function of time. Thus the epicycle model should be able to produce very much the same phenomena as the eccentric model.

Also the epicycle model has an apogee A where P has its maximum distance (ρ + r) from T. Accordingly A is the point where P passes the most distant point of the epicycle. In a similar way there is a perigee Π with a minimum distance (ρ − r) placed at the nearest point of the epicycle. Whether A and Π are diametrically opposite points depends on the ratio between $\omega_{\odot}$ and Ω.

The epicycle model can be shaped in two different forms according to the sign of Ω. If Ω has the same sign as $\omega_{\odot}$, then the motion of P on the epicycle will take place in the same direction as the motion of the epicycle centre C on the deferent (cf. Figure 5.3 a), that is from West to East, or *secundum ordinem signorum*, as the Latin astronomers said. We shall call this particular case the direct epicycle model.

If Ω has the opposite sign to $\omega_{\odot}$, then the motion of P on the epicycle will be from East to West (cf. Figure 5.3 b), or *contra ordinem signorum*. We shall call this variant the indirect epicycle model.

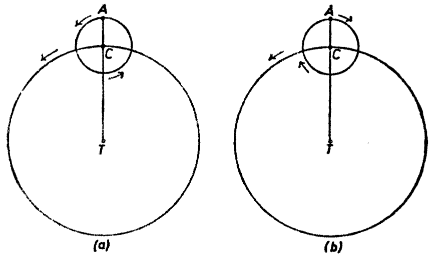


Fig. 5.3.

One characteristic difference between the two cases appears when we consider the apparent velocity of P* as seen from the Earth T. It follows immediately from Figures 5.3 a–b that the maximum apparent velocity (daily motion) of P* will occur at the apogee of the direct, but at the perigee of the indirect model [III, 3; Hei 1, 218].

The Equivalent Models

The inventor of the epicyclic model is usually thought to be Apollonius of Perga (about 200 B.C.) since, according to Ptolemy [XII, 1; Hei 2, 450], he found one of its characteristic properties, viz. that it can be so constructed that retrograde, apparent motions are produced (see page 331; cf. Neugebauer 1959 a, and Toomer 1970). In any case it is certain that it is older than Hipparchus, since the latter comments upon it. According to the commentator Theon of Smyrna (about A.D. 130) *Hipparchus declared it a worthy subject of research for the mathematicians to find the reason why so different hypotheses as, on the one hand, that which makes use of eccentric circles, and on the other that which applies concentric circles and epicycles, seem to produce the same effects* (*Lib. de Astron.* Ch. 26, p. 244 f.). Such an investigation is carried out in the Almagest [III, 3; Hei 1, 219] with the result that the two models will simulate precisely the same apparent motion under certain conditions which we can find from (5.3) and (5.4). In fact, the position vector $\overrightarrow{TP}$ of the Sun will have the same direction in the two models if

$$\overrightarrow{DP} = k \cdot \overrightarrow{TC} \quad \text{and} \quad \overrightarrow{CP} = k \cdot \overrightarrow{TD}$$

where k is the same number in both relations (cf. Proclus, *Hyp.* III, 3, p. 65). This implies

1) that the epicycle vector $\overrightarrow{CP}$ has the same fixed direction as the eccentricity vector $\overrightarrow{TD}$ from which follows that the Sun P performs a retrograde motion on the epicycle with an angular velocity $\Omega = -\omega_\odot$ which is numerically the same as it has on the eccentric circle, and

2) that the radius $R = |\overrightarrow{DP}|$ of the eccentric circle has the same ratio k to the radius $\rho = |\overrightarrow{TC}|$ of the concentric deferent as the radius $r = |\overrightarrow{CP}|$ of the epicycle has to the eccentricity $e = |\overrightarrow{TD}|$, or

$$\frac{R}{\rho} = \frac{r}{e}$$

If we choose the models so that $r = \rho$ we have

$$R \cdot e = r^2$$

which entails the possibility of passing from one model to the other by a geometrical inversion (Neugebauer 1959 a, p. 8). This consequence will be used in the theory of retrograde motions (see page 334).

If we put $k = 1$ we have $R = \rho$ and $r = e$; in this case the distance of the apogee (respectively the perigee) from the Earth will be the same in the two models to which we shall refer as the equivalent models.

The equivalence implies that the properties of any of the two models may be investigated from each of them with exactly the same result. In most of the following cases we shall examine the eccentric model only, although Ptolemy is careful to prove all his statements first from one model and then from the other.

The Variables of the Eccentric Model

Before investigating the theory of the eccentric model more closely it will be convenient to define a number of variables which, as coordinates, determine the configuration of the model at any given time t (cf. Figure 5.4). We have already introduced $\lambda_m(t)$ by (5.2) as the mean longitude of the Sun. This coordinate was called the *medius motus* by Latin astronomers, a term which is best rendered as mean motus. It is represented by the arc from the vernal equinox ♈ to the point M* on the ecliptic denoting the mean Sun, i.e.

$$\lambda_m(t) = \text{angle } ♈TM^* \tag{5.5}$$

In the same way the true longitude $\lambda(t)$ of the Sun is defined as

$$\lambda(t) = \text{angle } ♈TP^* \tag{5.6}$$

and is represented by the arc from ♈ to the point P* denoting the apparent position of

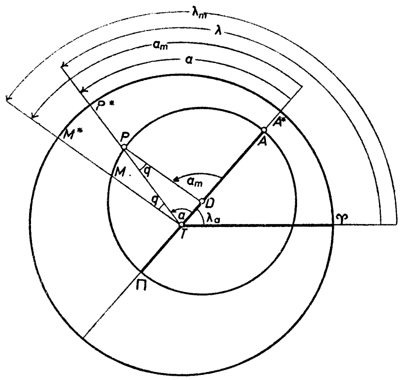


Fig. 5.4.

the Sun on the ecliptic. Latin astronomers called this arc the *verus locus* of the Sun, or – if parallax was taken into account – the *locus apparens.*

In general, Ptolemy makes use of longitude co-ordinates only when discussing observations. In more theoretical investigations he prefers another set of co-ordinates, referring to the apsidal line instead of to the vernal equinox. Thus he introduces the variable

$$a(t) = \text{angle } A^*TP^* = \text{angle } ATP \tag{5.7}$$

which measures the angular distance of the apparent Sun P* from the apsidal line TA* as seen from the Earth. Since the position vector $\overrightarrow{TP}$ rotates about T with a non-uniform velocity, a(t) is an increasing, non-linear function of time. It is called the anomaly, or more precisely the true anomaly of the Sun. In the Almagest the term anomaly is used in more than one sense[9]). Usually it means a deviation from some kind of uniform motion, but here it is applied to the non-linear function (5.7) itself. Later Latin astronomers avoided the ambiguity, calling a(t) the *argumentum verum* of the Sun, a term which we shall translate as true argument.

The other variable belonging to this set is the mean anomaly, or *argumentum medium* (the mean argument) defined by

$$a_m(t) = \text{angle } ADP \tag{5.8}$$

It defines the position of the real Sun P in its eccentric orbit. Since the deferent vector $\overrightarrow{DP}$ rotates with constant angular velocity, $a_m(t)$ is an increasing, linear function of time.

9) In his *Epitome Astronomiae Copernicanae* (V, 2, 3, ed. Caspar p. 386) Kepler explains that *although, properly speaking, anomaly – irregularity – is a change in the movement of the planet, nevertheless astronomers employ this word for the very motion in which the irregularity is present.* In the Almagest the word anomaly is used in so many different meanings that one cannot but sympathise with Rome (1943, p. 142) when he says that one never ought to use this ambiguous term in connection with ancient astronomy. One of the principal achievements of Mediaeval Latin astronomy was to create a more precise terminology with different terms for different concepts.

We must now consider the relations between the mean and the true longitude, and between the mean and true argument. In the epicycle model (Figure 5.2) the vector $\overrightarrow{TC}$ rotates so that it always points towards the position M* of the ecliptic mean Sun. Also in the eccentric model (Figure 5.4) the motion of the mean Sun M* is coupled to the motion of the true Sun P in such a way that the apparent Sun P* will never be far away from M*. This coupling is established in such a way that the mean motus vector $\overrightarrow{TM^*}$ is always parallel to the deferent vector $\overrightarrow{DP}$; this is possible because they rotate with the same angular velocity. We can then define

$$q = \text{angle } M^*TP = \text{angle } TPD \tag{5.9}$$

It is seen from Figure 5.4 that the angle q can be expressed as the difference between the true and mean longitudes

$$q = \lambda(t) - \lambda_m(t) \tag{5.10}$$

or as the difference between the true and mean arguments

$$q = a(t) - a_m(t) \tag{5.11}$$

Re-arranging the terms we find

$$\lambda(t) = \lambda_m(t) + q \tag{5.12}$$

and

$$a(t) = a_m(t) + q \tag{5.13}$$

which shows that q can be conceived as a correction which – added to the mean coordinate – produces the corresponding true coordinate.

The coupling between the motions of M* and P* implies that either of the equivalent models has only one degree of freedom in the sense that the deviation from uniform, circular motion can be described by a function q of one independent variable which may be either the mean or the true argument. If we consider q as a function q(a) of the true argument it satisfies the symmetry relation

$$q(a) = -q(a + 180°) \tag{5.14}$$

which is carefully proved in the Almagest [III, 3; Hei 1, 230].

A model of this kind with only one degree of freedom is said to have a s i n g l e a n o m a l y. Already here Ptolemy remarks [III, 3; Hei, 219] that the planets exhibit a double anomaly which can be accounted for by a combination of the eccentric and the epicycle model (see page 267).

Ptolemy called q(a) the *difference of anomaly* in agreement with (5.11), or the *prosthaphairesis* of anomaly. The latter word was probably coined by himself from the Greek verbs *prostithemi* (to add) and *aphaireo* (to subtract) as a term denoting a quantity which has to be added or subtracted. This is understandable when we re-

member that negative numbers were unknown, so that e.g. (5.13) had to be understood as

$$a(t) = a_m(t) \pm q$$

where q itself is a positive quantity to be added to or subtracted from $a_m(t)$. In Latin astronomy q(a) was called the *equatio centri,* or the equation of centre; this term is derived from the Latin verb *equare* (to equate) since a(t) and $a_m(t)$ may be 'equated' by addition or subtraction of q(a).

In the following we shall consider q(a) as essentially an additive correction in accordance with (5.12–13), and give it the appropriate sign, viz.

$$q(a) < 0 \quad \text{for} \quad 0° < a < 180° \tag{5.15}$$

$$q(a) > 0 \quad \text{for} \ 180° < a < 360° \tag{5.16}$$

Finally we notice two further relations between the variables, viz.

$$\lambda_m(t) = a_m(t) + \lambda_a \tag{5.17}$$

$$\lambda(t) = a(t) + \lambda_a \tag{5.18}$$

where λ_a is the longitude of the apogee A*. They are easily proved from Figure 5.4.

The Maximum Value of q(a)

Ptolemy now investigates the problem of the maximum value of q(a). It is easily solved trigonometrically by means of the sine relations applied to the triangle TDP (Figure 5.4), giving

$$\sin q = \frac{e}{R} \sin a \tag{5.19}$$

from which follows that we have a maximum value of $q(a_m)$ determinated by

$$\sin q_{max} = \frac{e}{R} \quad \text{for} \quad \sin a = 1 \tag{5.20}$$

This means that q is maximum at the two positions of the Sun P_1 and P_2 with the true anomaly $a = 90°$ or $270°$ respectively, in agreement with the result found by Ptolemy. These two positions were called the *longitudines mediae* in Mediaeval Latin astronomy. Geometrically they are determined by a line through T perpendicular to the apsidal line.

Ptolemy's proof [III, 3; Hei 1, 222] is geometrical and given in relation to Figure 5.5, in which we have retained the notation found in the Almagest. Thus Z is the centre of the Earth and E the centre of the deferent; B and Δ are the points called P_1

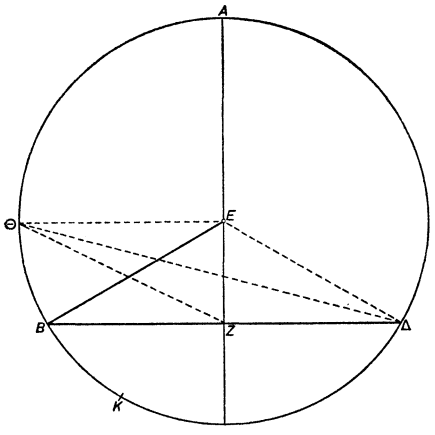


Fig. 5.5.

and P_2 above; Θ is a point for which the mean anomaly is smaller than for B. We then have to prove that

$$\text{angle EBZ} = q(B) > q(\Theta) = \text{angle E}\Theta\text{Z}$$

The proof is founded on a proposition in Euclid III, 7 which states that if Z is a point on the diameter through A outside the centre E of the circle, then the distance ZΘ will increase steadily as Θ recedes from A.

This means that in the triangle ΘΔZ we have

$$\Theta Z > Z\Delta \; (= ZB)$$

and therefore

$$\text{angle } \Theta\Delta Z > \text{angle } \Delta\Theta Z$$

We also have

$$\text{angle E}\Delta\Theta = \text{angle E}\Theta\Delta$$

so that by addition we get

$$\text{angle E}\Delta\text{Z} > \text{angle E}\Theta\text{Z}$$

But

$$\text{angle E}\Delta\text{Z} = \text{angle EBZ}$$

so that

$$\text{angle EBZ} > \text{angle E}\Theta\text{Z}$$

which was to be proved.

In a similar way Ptolemy is able to prove that if K is a point with a greater mean anomaly than B we have

angle EBZ > angle EKZ

This means that the maximal angle q(a) is at B, or Δ.

This is a beautiful example of how Ptolemy was able to determine extremal properties of a function without having recourse to mathematical analysis (cf. page 89).

General Theorems of Motion

Ptolemy further maintains [III, 3; Hei 1, 220] that at the *longitudines mediae* P_1 and P_2 the apparent angular velocity of the Sun (i.e. the angular velocity of P) will be equal to its mean angular velocity $\omega_\odot$ determined by (5.1). Thus P_1 and P_2 may be called the points of mean motion. This follows from (5.12) which gives

$$\frac{d\lambda}{dt} = \frac{d\lambda_m}{dt} + \frac{dq}{dt}$$

Here the left hand term is the angular velocity of P*. On the right hand side the first term is equal to $\omega_\odot$, which follows from (5.2). The second term is zero at the points where q has a maximum value.
Thus

$$\frac{d\lambda}{dt} = \omega_\odot \quad \text{for} \quad a = 90° \text{ or } 270° \tag{5.21}$$

in agreement with Ptolemy's assertion, of which there is no rigorous proof in the Almagest. It is possible that Ptolemy saw the truth of this statement in an intuitive way, perhaps from the prosthaphairesis table mentioned below (page 151). Actually, this table shows that q(a) has a negligible variation around the maximum positions P_1 and P_2, so that we here must have the same rate of change of a(t) and $a_m(t)$.

Another general theorem is concerned with the times in which P goes through certain parts of the eccentric circle. Let P pass the apogee A, the point of mean motion P_1, and the perigee Π at the points of time t(A), $t(P_1)$ and t(Π) respectively. Since the motion is uniform, we have according to Figure 5.6

$$\frac{t(P_1) - t(A)}{t(\Pi) - t(P_1)} = \frac{\text{angle } ADP_1}{\text{angle } P_1D\Pi} = \frac{90° + |q_{max}|}{90° - |q_{max}|} > 1$$

or

$$t(P_1) - t(A) > t(\Pi) - t(P_1) \tag{5.22}$$

Expressed in words (5.22) states that in the equivalent models the time used by P to go from the apogee to the point of mean motion is greater than the time from the latter point to the perigee [III, 3; Hei 1, 220].

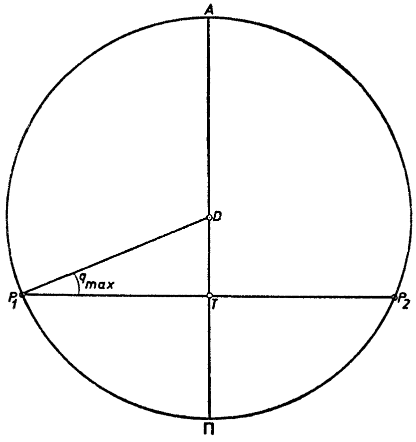


Fig. 5.6.

The importance of this proposition becomes clear if we assume that the three points of time in (5.22) can be found directly from observation. The inference is that in case such observations did not satisfy (5.22), the motion of P could not be described by the equivalent models. Thus (5.22) becomes a criterion for the applicability of these models (cf. page 268).

The Choice of the Solar Model

After this long examination of the general eccentric and epicycle models Ptolemy has now to decide which model he is going to use for his theory of the Sun's motion. His choice is based on the following presuppositions:

1) That the Sun has a single anomaly [III, 4; Hei 1, 232]. This is stated as a known fact, but strictly speaking we cannot yet be sure that the motion of the Sun can be described by a model with one degree of freedom. In fact, this has to be proved through the investigation below. That Ptolemy seems to presuppose what he has to prove is most easily explained from the fact that the solar theory is not due to himself, but to Hipparchus, so that the following treatment is no original piece of research, but only a pedagogical exposition.

2) That the Sun's motion satisfies the condition (5.22) [ibid.]. This is said to be in agreement with the phenomena; but no precise observations are referred to.

Taken together, 1) and 2) make it plausible that the theory of the Sun may be constructed by means of any one of the equivalent models described above. However, Ptolemy finally decides upon the eccentric model. The reason for making this choice is of course not founded upon any empirical argument, since the equivalence of the

models makes them equally well adapted to reproduce observable phenomena. But Ptolemy prefers the eccentric model because it is more simple from a theoretical point of view. Logically it would seem more correct to stick to the eccentric hypothesis since this achieves its purpose with one movement, not with two. This shows that it is the purely epistemological principle of economy which is the deciding factor in the choice between two equivalent mathematical descriptions of the same empirical material, and that no physical considerations are involved.

The Empirical Basis of the Solar Theory

The problem is now whether it is possible to provide the eccentric model with such numerical values of the parameters R, e, and λ_a that it will simulate the observed apparent motion of the Sun. This is the subject matter of a particular chapter [III, 4; Hei 1, 232] in which Ptolemy shows how such values can be found by means of the observed lengths of the seasons, which we shall denote by J_1 (spring), J_2 (summer), J_3 (autumn) and J_4 (winter).

He begins by relating that Hipparchus found

$$J_1 = 94\tfrac{1}{2}^d$$
$$J_2 = 92\tfrac{1}{2}^d$$

But since these values were determined almost 300 years before Ptolemy's time he wants to check them by means of some observations of his own made in A.D. 139–149, viz. the observations S_{13}, S_{14} and S_{15} in the table above (page 130).

Denoting the times of the three observations by t(13), t(14) and t(15) we have for the length of the spring season in A.D. 140

$$J_1 = t(15) - t(14) = 94^d13^h$$

This is according to the table of observations; but Ptolemy does not hesitate to round this off to the Hipparchian value $94\tfrac{1}{2}^d$. For the interval between the autumnal equinox A.D. 139 and the vernal equinox A.D. 140 we find from the table

$$J_3 + J_4 = t(14) - t(13) = 178^d6^h$$

in agreement with the $178\tfrac{1}{4}^d$ given by Ptolemy and Hipparchus. By means of the round value $365\tfrac{1}{4}^d$ of the length of the tropical year (cf. page 131). Ptolemy finally computes the length of the summer season A.D. 140 as

$$J_2 = 365\tfrac{1}{4} - (178\tfrac{1}{4} + 94\tfrac{1}{2}) = 92\tfrac{1}{2}^d$$

Again this is in accordance with Hipparchus. These values of J_1 and J_2 are the data upon which the following computation is based. We notice that if Ptolemy had used the Hipparchian value $365^d;14,48$ of the year and his own value of $J_1 = 94^d13^h = 94^d;32,30$, the resulting value of J_2 would have been $J_2 = 92^d;27,18$.

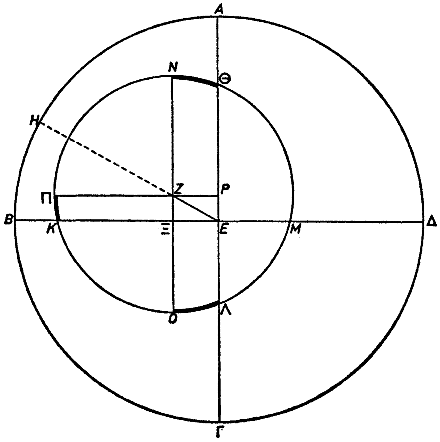


Fig. 5.7.

Computation of the Parameters

The following Figure 5.7 – in which we have retained Ptolemy's notation – shows the ecliptic and the eccentric circle with their respective centres E and Z. At the vernal equinox the Sun seems to be at ♈ = A, but it is really at the point Θ on the eccentric orbit. Similarly its position B on the ecliptic at summer solstice corresponds to K on the orbit, and the position Γ at the autumnal equinox corresponds to Λ. The corresponding arcs are easily found to be

$$\begin{array}{lll} \Theta K & = 94\tfrac{1}{2} \cdot \omega_{\odot} & = 93°;\ 9 \\ K\Lambda & = 92\tfrac{1}{2} \cdot \omega_{\odot} & = 91°;11 \\ \Theta K + K\Lambda & = & 184°;20 \\ \Lambda M\Theta & = & 175°;40 \end{array}$$

where $\omega_{\odot}$ is given by (5.1). The problem is, therefore, to place the eccentric circle inside the ecliptic in such a way that these three arcs are projected from the Earth as arcs of 90°, 90° and 180° on the ecliptic. This is a special case of a problem which we shall meet later in other connections (see page 172 and 273).

Since spring is the longest season it is clear that the 'longest quarter' of the eccentric must be placed in the first quadrant AEB in the figure. Through Z we draw diameters NO parallel with AΓ, and PΠ parallel with BΔ. We can then compute the equal arcs NΘ and OΛ from

$$N\Theta + 180° + O\Lambda = \Theta K + K\Lambda = 184°;20$$

from which follows

$$N\Theta = O\Lambda = 2°;10$$

For the arc ΠK we find similarly

$$\Pi K = 93°;9 - N\Theta - 90° = 0°;59$$

It is now possible to compute

$$PZ = 60^p \cdot \sin 2°;10 = 2^p;16 \text{ and}$$

$$PE = 60^p \cdot \sin 0°;59 = 1^p;2$$

where the radius of the deferent has been given the arbitrary value $R = 60^p$, presumably to make it easier to use the Table of Chords (see page 56).

It is now easy to find the eccentricity and the apogee of the eccentric circle. By the theorem of Pythagoras we have

$$e = EZ = \sqrt{PZ^2 + PE^2} = 2^p;29,30$$

which is very nearly $60^p : 24$. For the eccentricity we have accordingly

$$e/R = 1/24 \qquad (5.23)$$

Furthermore

$$\sin \lambda_a = \sin ZEP = \frac{PZ}{ZE} = \frac{2^p;16}{2^p;29,30}$$

from which follows

$$\lambda_a = 65°;30 \qquad (5.24)$$

as the ecliptic longitude of the apogee of the eccentric circle. Modern computations show that in Hipparchus' time the true value was $\lambda_a = 66°;14$. As we shall see (page 242, 253 and 257) this erroneous apogee entails an error of about $\frac{3}{4}°$ not only in the theory of the Sun, but also in the longitudes of the planets and the fixed stars, in cases where the positions of the latter are determined relative to a planet.

The problem of situating the eccentric is then solved, but it cannot be denied that the method calls for some comments.

The Errors in e and λ_a

Ptolemy himself made only one comment on the preceding determination of the longitude of the apogee of the Sun, viz. that he found the same value $\lambda_a = 65°;30$ as Hipparchus. Considering that he based his calculation on precisely the same data it is but little wonder that he arrived at the same results. But this led him and other ancient astronomers such as Proclus (*Hyp.* III, 3, p. 73) to believe not only that the solar apogee had kept its position unchanged since Hipparchus' time, but also that it would have the same ecliptic longitude forever [III, 4; Hei 1, 233]. Nevertheless, we now know that this is erroneous, since apart from precession the apsidal line of the orbit of the Earth has a slow eastward motion of ca. 12″ per year[10]).

10) Almost all histories of astronomy have ascribed the discovery of the proper motion of the apogee of the Sun to the 9th century Muslim astronomer al-Battānī; but it seems that the first who clearly distinguished this motion from precession was al-Bīrūnī (ca. A.D. 1000), while the Toledo astronomer al-Zarqālī (ca. 1050) was the first to assign a definite value to it. – See Hartner and Schramm (1963), and Toomer (1969).

This means that in the 300 years between Hipparchus and Ptolemy the solar apogee must have moved more than 5° towards the East. If Hipparchus' value $\lambda_a = 65°;30$ were corrected then at Ptolemy's time we should have expected λ_a to be ca. 70°.

The question why Ptolemy did not discover this motion has two different aspects. One is his use of round values for the length of the tropical year and the spring season. Here he obviously wished to stick to the tradition of Hipparchus, in whose observations he had a very firm belief. Nevertheless, one cannot help wondering why Ptolemy did not take the trouble to investigate the influence of his approximations on the final values of the parameters.

This would have been possible without a general theory of errors[11]). Actually it meant no more than that he had had to do the same calculation as before, starting with the less approximate values (see page 133 and 145)

$$1^a = 365^d;14,48$$
$$J_1 = 94^d;32,30 \quad (= 94^d 13^h)$$
$$J_2 = 92^d;27,18$$

A calculation shows that the arcs ΘK and $K\Lambda$ would now be changed to

$$\Theta K = 93°;11,0$$
$$K\Lambda = 91°;7,38$$

This would cause a very slight change in the eccentricity to

$$EZ = 2^p;30,18$$

and a change in the longitude of the apogee to

$$\lambda_a = 64°;33$$

The result is that if Ptolemy had stuck to his own observations he would have found an insignificant change of the eccentricity, and a change of position of the apogee amounting to $\Delta\lambda_a = 57'$ to the West, compared with Hipparchus' values. Here $\Delta\lambda_a$ has the wrong sign; but it is also clear that its numerical value is so small that we must acquit Ptolemy's approximations as the primary cause of his erroneous result. We must then look elsewhere, and the most obvious hypothesis is that there are observational errors in Ptolemy, or in Hipparchus, or in both of them.

This question has been investigated by Rome (1943, p. 153 f.) and cleared in a fundamental paper by V. Petersen and O. Schmidt (1967), who compared the tradi-

11) Although a general theory of errors was unknown to Antiquity there are many testimonies to the fact that Hellenistic astronomers were aware both of several types of observational errors and of their influence on calculations. Ptolemy mentions unsatisfactory construction or mounting of instruments and quotes Hipparchus for an evaluation of the ensuing errors [III, 1; Hei 1, 194 f.]. Theon of Alexandria (ed. Rome, pp. 819 ff.) gives an example in which he assumes the graduated ring of a meridian instrument to be correctly placed in the plane of the meridian, but so that there is an error of 6 minutes of arc in the measured zenith distances. He then determines the resulting error of the declination and longitude of the Sun, and of the equinox. This is a rare example, and many speculations on Ptolemy's good or bad faith could have been avoided if he had given a little more information on such questions. – See also Rome (1937, pp. 227 ff.).

tional values with values computed by means of modern tables and found that J_1 was about 12^h too long in Hipparchus and $14^h;24$ too long in Ptolemy. J_2 was about 4^h too long in Hipparchus and $1^h;30$ too short in Ptolemy. But a computation shows that an error of $\pm 6^h$ in J_1 and J_2 produces an error of about 7° in λ_a. In other words, both sets of observations are too unreliable to make a satisfactory determination of λ_a possible. That Hipparchus found a value of λ_a deviating only about 0°;44 from the true value must accordingly be a result of mere chance.

Finally we notice that immediately after determining the parameters of the solar theory, Ptolemy uses it to find the lengths J_3 and J_4 of the remaining seasons. He confirms again Hipparchus' values $J_3 = 88\frac{1}{8}^d$ and $J_4 = 90\frac{1}{8}^d$; but he makes no attempt to compare them with observations. This looks like a bad understanding of scientific method, but is, in fact, not characteristic of the Almagest as a whole. The most natural explanation of this omission is that Ptolemy did not want to check his solar theory against the lengths of the seasons, but against the general course of the Sun during the year (see page 154).

The Size of the Eccentric Model

It appears from the derivation of the parameters of the eccentric model that Ptolemy is unable to find e and R separately, but only the ratio $e/R = \frac{1}{24}$. Thus the actual magnitude of both the eccentricity and the deferent radius remain undetermined. This must, in fact, be so since the only available data were times determining angles. The actual size of the model cannot be found except if at least one linear quantity, or distance, is known, e.g. the distance of the Sun in a given position. We know from the Almagest [V, 11; Hei 1, 402] that Ptolemy doubted whether the Sun has an observable parallax. He also states that Hipparchus assumed it to have a parallax just great enough to make it possible to find its distance – whatever that means –, and further that he used a solar eclipse to find the distance of the Moon. We shall return to this question in connection with Ptolemy's theory of eclipses (page 209). Here we recall that most of the Almagest is concerned with the determination of longitudes and latitudes only, so that it is sufficient to know the relative dimensions of each planetary model. This means that Ptolemy can give one of the linear parameters an arbitrary value and express the others accordingly, as when above he gave the deferent radius the arbitrary length $R = 60^p$ (p meaning *partes*) and found the eccentricity to be $e = 2^p;30$ in the same unit. In the *Planetary Hypotheses* Ptolemy made an attempt to determine absolute values of all the distances in the solar system, based not on empirical data, but on a hypothesis concerning the celestial spheres. This attempt is briefly described in the final chapter (page 393).

The Equation of Centre

After the parameters of the eccentric model have been found, only two problems remain before it is ready for use. The first is to calculate a table of the prosthaphairesis

angle, or the equation of centre. Since the mean argument can be found from the tables of mean motion it will be convenient to determine $q(a_m)$ as a function of this variable. From Figure 5.4 we see immediately that

$$q(0°) = q(180°) = 0$$

The maximum value is found from (5.20) and (5.23):

$$q_{max} = 2°;23 \tag{5.25}$$

According to (5.20) it occurs for $a(t) = 90°$ or $270°$, that is for $a_m(t) = 92°;23$ or $267°;37$.

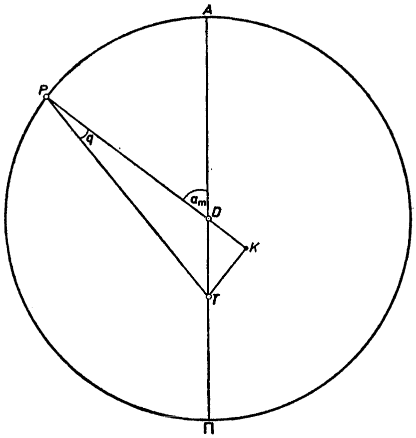


Fig. 5.8.

For an arbitrary value of a_m the equation of centre can be found from Figure 5.8, showing the eccentric model with the usual notation. The point K is determined by the perpendicular from T on the prolongation of PD. We then have [III, 5; Hei 1, 240]

$$TK = e \cdot \sin a_m$$
$$KD = e \cdot \cos a_m$$
$$KP = e \cdot \cos a_m + R$$

The prosthaphairesis angle can now be found from

$$\sin q(a_m) = \frac{TK}{TP} = -\frac{e \cdot \sin a_m}{\Delta(a_m)} \tag{5.26}$$

where

$$\Delta(a_m) = TP = \sqrt{e^2 \sin^2 a_m + (e \cos a_m + R)^2} \tag{5.27}$$

is the distance of the centre P of the Sun from the centre T of the Earth, corresponding to the mean argument a_m. The expression (5.26) is equivalent to the simpler formula

$$\tan q(a_m) = -\frac{e \sin a_m}{R + e \cos a_m} \tag{5.28}$$

In (5.26) and (5.28) the negative sign means that the relations (5.15–16) are satisfied.

Formulae of this kind could not be used by Ptolemy and only illustrate the general procedure of his computation of various values of $q(a_m)$. Neither is the result given as a general functional relation like (5.26), but in the form of a *Table of Equations* [III, 6; Hei 1, 253] in which values of $q(a_m)$ are tabulated in degrees and minutes corresponding to the arguments

$$a_m = 6°, 12°, \ldots, 90°, 93°, 96°, \ldots 177°, 180°$$

Because of the symmetry relation (5.14) this is sufficient to cover a complete revolution. For arguments not listed in the table one has to determine $q(a_m)$ by linear interpolation. In the *Handy Tables* there is a similar table of equations with intervals of 1° only.

The Epoch of the Solar Theory

By means of (5.13) and (5.26) we are able to find the true argument of the Sun at a time when the mean argument is known. The only remaining problem is to determine the mean argument $a_m(t_0)$ at a definite point of time t_0, called the **epoch**. For t_0 Ptolemy usually chooses the beginning of the era of Nabonassar (cf. page 127)[12]. In the solar theory this problem is solved by means of the observation S_{12} (see the list page 130) of an autumnal equinox, an observation made by Ptolemy himself.

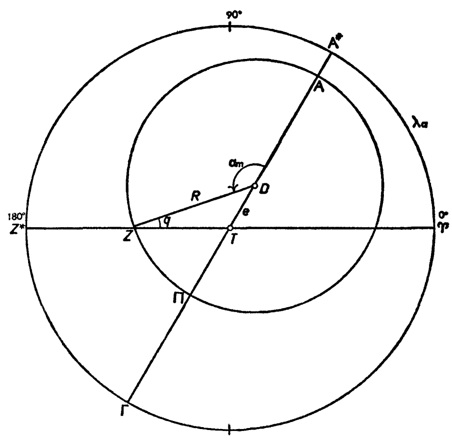


Fig. 5.9

12) It is worth noticing that the Almagest shows no trace of any considerations of what might be called the initial conditions of the universe. Since Ptolemy believed in the eternity of celestial motions [I, 1; Hei 1, 6] such questions could not arise. Later both Muslim and Christian astrologers – believing the world to be created in time, or together with time – were much interested in determining the date of 'great conjunction' of all the planets marking the event of the creation. This was an influence of older, Persian ideas of a cyclic life of the universe. – See Kennedy (1962).

First Ptolemy calculates the mean argument of the Sun at the autumnal equinox, denoted by Z (or Z*) in Figure 5.9. The sine relations applied to the triangle DZT give

$$\sin q(Z) = \frac{e \sin (180° - \lambda_a)}{R}$$

which leads to

$$q(Z) = 2°;10$$

as the numerical value of the equation of centre at the autumnal equinox. Since

$$\text{angle ZDT} = \text{angle ZT}\Pi - q(Z) = \lambda_a - q(Z)$$

we have

$$\text{angle ZDT} = 65°;30 - 2°;10 = 63°;20$$

This means that at the autumnal equinox the Sun is always 63°;20 before its perigee, that is 116°;40 behind its apogee. Thus we have in general

$$a_m(Z) = 116°;40$$

at the autumnal equinox.

The assumption of a fixed position of the apogee relative to the ecliptic (see page 147) means that this number must be regarded as a constant independent of time. Therefore the observation S_{12} tells us that also at the time t_{12} (= Hadrian 17 Athyr 7, 2^h after noon) the mean argument of the Sun was $a_m(t_{12}) = 116°;40$. Its value $a_m(t_0)$ at the beginning of the era of Nabonassar can now be found by the tables of mean motion. We know from page 127 that there are 879 Egyptian years from Nabonassar 1 Thoth 1 to Hadrian 17 Thoth 1, and that there are furthermore 66 days from Thoth 1 to the beginning af Athyr 7. Since Ptolemy here reckons the natural day from noon, there are still two hours left. The calculation of the mean motion of the Sun during this time is then carried out in the following way:

Mean motion in 810^a =	163°; 4,12	
Mean motion in 54^a =	346°;52,16	
Mean motion in 15^a =	356°;21,11	
Mean motion in 879^a =	....	146°;17,39
Mean motion in 60^d =	....	59°; 8,17
Mean motion in 6^d =	....	5°;54,49
Mean motion in 2^h =	....	0°; 4,55
		211°;25,40

This means that from the beginning of the era of Nabonassar to the time of the observation the Sun has moved 211°;25,40 towards a position with the mean argument $a_m(t_{12}) = 116°;40$. Therefore, the mean argument at the time t_0 is

$$a_m(t_0) = a_m(t_{12}) - 211°;25,40$$

or

$$a_m(t_0) = 265°;14,20 \tag{5.29}$$

Also the mean ecliptic longitude $\lambda_m(t_0)$ at this epoch can be found from (5.29) and (5.17); this gives a value which Ptolemy rounds off to

$$\lambda_m(t_0) = 330°;45 \tag{5.30}$$

The epochal values $a_m(t_0)$ and $\lambda_m(t_0)$ are listed at the head of the Tables of Mean Motion [III, 2; Hei 1, 210]. In Mediaeval Latin astronomy they were called the *radices* (sing. *radix*) of a_m and λ_m respectively corresponding to the era of Nabonassar.

The Final Formula of the Solar Theory

The solar theory is now complete, and in a special chapter [III, 8; Hei 1, 257] Ptolemy explains the procedure for computing the position of the Sun at a given moment of time t. It comprises the following operations

1) Calculate the time $(t - t_0)$ elapsed from the epoch t_0 to the time t of the calculation.

2) Enter $(t - t_0)$ as argument into the tables of mean motion, and find the product $\omega_\odot(t - t_0)$.

3) Add the *radix* $a_m(t_0)$ of the mean argument in order to find

$$a_m(t) = a_m(t_0) + \omega_\odot(t - t_0) \tag{5.31}$$

as the mean argument at the time t.

4) Enter the table of equations of centre with $a_m(t)$ and find the equation $q(a_m)$.

Now the procedure splits into two possible steps. If we want to express the position of the Sun by its true argument we simply have to

5 a) Add the equation $q(a_m)$ to the mean argument $a_m(t)$. This gives $a(t)$ according to (5.13), which may be combined with (5.31) to give the final formula

$$a(t) = a_m(t_0) + \omega_\odot(t - t_0) + q(a_m) \tag{5.32}$$

On the other hand we may express the position by the true longitude of the Sun. This implies that we

5 b) Add $\omega_\odot(t - t_0)$ to the *radix* $\lambda_m(t_0)$ of the mean longitude, which gives us $\lambda_m(t)$ according to (5.2). Then add the equation $q(a_m)$ to $\lambda_m(t)$ according to (5.12).

The latter procedure corresponds to the general formula

$$\lambda(t) = \lambda_m(t_0) + \omega_\odot(t - t_0) + q(a_m) \tag{5.33}$$

In this way the problem of finding the true position of the Sun is solved without more complicated operations than simple additions and subtractions of numbers found in ready-made tables[13]). It is true that the use of these tables may involve linear inter-

13) We notice that the Almagest gives only a procedure for finding the true longitude or argument of the Sun by the tables of mean motions and equations. There is no procedure for the inverse problem of finding the mean from the true position (cf. Kepler's problem), but this question turns up later in the theory of latitudes (see page 367).

polation, and consequently multiplication of sexagesimal numbers. But the great advantage of the theory is that no trigonometrical calculations are left – they have all been made by Ptolemy, and their results stored in the table of equations. In the following chapters we shall see how Ptolemy also succeeds in bringing the other planetary theories into a similar form, so that they could be handled by simple arithmetical calculations without trigonometry.

The final chapter in Book III begins with the words *We have now finished that part of the theory which is concerned with the Sun alone* [III, 9; Hei 1, 258]. This is true in so far as we are now able to compute the apparent (observable) position of the Sun at any given time. But one wonders a little that Ptolemy takes no steps to verify the theory, comparing a number of computed positions with observations, at least throughout a period of one year. Ptolemy seems to have been convinced of the correctness of the theory for the reason that it satisfies the few observations used to develop it. However, this latter comment is not quite fair. The solar theory is important, not only because it describes the motion of the most conspicuous heavenly body, but even more because it is an essential element of the theories of all the other planets. This means that the theory of Sun receives an indirect verification each time one of the planetary theories is confirmed.

The Equation of Time

Only one particular application of the solar theory is mentioned in Book III, the last chapter of which [III, 9; Hei 1, 258] is concerned with the inequality of the solar day, that is with what Latin astronomers called the *equatio dierum*, or the equation of time. The problem arises from the fact that the term 'one day' has several meanings in astronomical terminology. Thus Ptolemy mentions

1) the period d_s, now called one sidereal (or stellar) day, in which the universe completes one rotation about the poles of the equator. More precisely we defined it as the interval of time between two successive culminations (or risings) of the same point on the equator. It is an astronomical constant because the diurnal motion of the heavens is taken to be uniform.

2) The period d(t) between two successive culminations of the centre of the Sun S* is called one true solar day. It is the same as the interval between successive culminations of the point S upon which S* is projected upon the equator (see Figure 5.10). Since the Sun has a proper motion towards the East along the ecliptic of about 1° per day, d(t) will be about 4 minutes of time longer than d_s.

The true solar day d(t) is no astronomical constant but a function of time. It varies for two reasons: a) The daily motion of the Sun S* varies throughout the year, having its maximum value at the perigee, its minimum at the apogee, and its mean value at the *longitudines mediae* (see above, page 141). In Figure 5.10 this is illustrated by the varying distances of successive positions of the Sun on the ecliptic. b) But even if S*

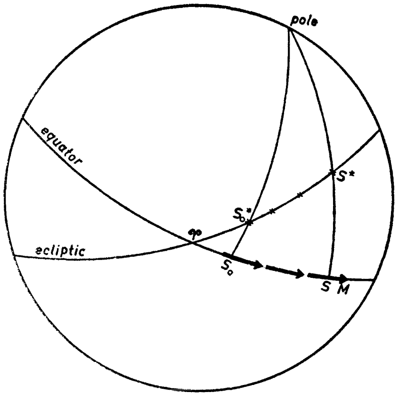


Fig. 5.10

had had a uniform motion along the ecliptic it would not pass the meridian at equal intervals of time. For because of the obliquity of the ecliptic, equal arcs on the latter are not projected as equal arcs on the equator, so that a uniform motion of S* would give a non-uniform motion of S. Therefore it is convenient to define

3) one mean solar day d_m as the interval between two successive culminations of a point M moving towards the East on the equator with a constant velocity, equal to the mean angular velocity of the Sun. This point is called the mean Sun by modern astronomers. It is not to be confused with the ecliptical mean Sun M* defined above (see p. 133). We shall call it the equatoreal mean Sun. It follows from this definition that d_m is an astronomical constant.

We can now define true solar time as the hour angle h(S*) of S*, and mean solar time as the hour angle h(M) of the equatoreal mean Sun M, in both cases reckoning from noon, that is from upper culmination. In modern astronomy the difference

$$E = h(S^*) - h(M) \tag{5.34}$$

is called the equation of time. It can be visualized as the difference between the time shown by a sundial and that shown by a chronometer adjusted to mean solar time. Since the latter is the basis of modern civil time the equation of time plays no great role in modern time reckoning.

In Ptolemaic astronomy the problem was both more serious and of a different character, because theoretical astronomy used a time reckoning different from that of the civil usage. Thus the tables of mean motion of the Sun (see above page 133) and the other planets were constructed with equal intervals of time between successive entries; in particular the table of days has equal intervals of one mean solar day d_m. On the other hand, in most observations of celestial phenomena the hour of the observation is given in true solar time, e.g. 1^h after noon, 2^h before sunrise, etc. The problem of the equation of time can now be defined in the following way (cf. Rome 1939, and Neugebauer 1962, p. 63 f.).

Let us consider an astronomical event A and a later event B for which we have (cf. Figure 5.10)

	A	B
Position of the true Sun	S_0^*	S^*
Position of the ecliptical mean Sun.	M_0^*	M^*
Position of the equatoreal mean Sun	M_0	M
Civil (true solar) time	$\tau_0 = h(S_0^*)$	$\tau = h(S^*)$
Mean solar time	$t_0 = h(M_0)$	$t = h(M)$
True longitude of the Sun.	$\lambda(\tau_0)$	$\lambda(\tau)$
Mean longitude of the Sun.	$\lambda_m(\tau_0)$	$\lambda_m(\tau)$
Right ascension of the Sun.	$\alpha(\tau_0)$	$\alpha(\tau)$
Right ascension of the equatoreal mean Sun . .	$\alpha_m(t_0)$	$\alpha_m(t)$

In civil time A and B are separated by a number of days equal to $(\tau - \tau_0)$ and in mean solar time by a number equal to $(t - t_0)$. Then the relation

$$t - t_0 = (\tau - \tau_0) + E \tag{5.35 a}$$

or

$$E = (t - t_0) - (\tau - \tau_0) \tag{5.35 b}$$

defines the equation of time E. It can be conceived as a correction which, added to the length of the interval of time between A and B, as measured by the motion of S^*, gives its length as measured by the motion of M. Accordingly, if E, τ, and τ_0 are known we can find the argument $(t - t_0)$ by the tables of mean motion of the Sun.

In (5.35) the term $(\tau - \tau_0)$ is equivalent to the arc

$$(\tau - \tau_0) \cdot 15^\circ = \alpha(\tau) - \alpha(\tau_0) \tag{5.36}$$

representing the increase of the right ascension of the true Sun S^* from A to B. Similarly the term

$$(t - t_0) \cdot 15^\circ = \alpha_m(t) - \alpha_m(t_0) \tag{5.37}$$

represents the increase of the right ascension of the equatoreal mean Sun M. Since by definition the latter completes a uniform revolution on the equator in exactly the same period in which the ecliptical mean Sun M^* completes a revolution on the ecliptic, it follows that the increase of the right ascension of M is equal to the increase of the longitude of M^*, that is the mean longitude $\lambda_m(t)$ of the Sun S^*. We have accordingly

$$\alpha_m(t) - \alpha_m(t_0) = \lambda_m(t) - \lambda_m(t_0) \tag{5.38}$$

This means that (5.37) can be found from the tables of mean motion of S^*.

We can now write (5.35) in the form

$$E = \frac{[\lambda_m(t) - \lambda_m(t_0)] - [\alpha(\tau) - \alpha(\tau_0)]}{15} \tag{5.39}$$

Let us now compare this formula with the procedure for determining E described very briefly in the Almagest, [III, 9; Hei 1, 262] where Ptolemy considers an interval $(\tau - \tau_0)$ using times reckoned from midnight or noon. It appears that the method is approximate only and comprises the following stages:

1° With the times τ and τ_0 we calculate

$$\Delta\lambda_m = \lambda_m(\tau) - \lambda_m(\tau_0)$$

and

$$\Delta\lambda = \lambda(\tau) - \lambda(\tau_0)$$

using the formulae (5.2) and (5.33) of the solar theory, with the argument $(\tau - \tau_0)$. This shows the approximate character of the procedure, since the tables had to be used with the argument $(t - t_0)$ which is still unknown. Ptolemy does not discuss how good the approximation is.

2° By means of the table of right ascensions we find

$$\Delta\alpha = \alpha(\tau) - \alpha(\tau_0)$$

starting with the longitudes $\lambda(\tau)$ and $\lambda(\tau_0)$ and using the method described above (page 153).

3° We then calculate

$$|E| = \frac{|\Delta\alpha - \Delta\lambda_m|}{15} \tag{5.40}$$

which is the Ptolemaic equation of time expressed as a non-negative function of τ and τ_0. It is seen that if we admit the approximation mentioned above then (5.40) is equivalent to (5.39) apart from its sign.

4° Finally we calculate

$$t - t_0 = (\tau - \tau_0) + |E| \quad \text{for} \quad \Delta\alpha > \Delta\lambda_m \tag{5.41 a}$$

or

$$t - t_0 = (\tau - \tau_0) - |E| \quad \text{for} \quad \Delta\alpha < \Delta\lambda_m \tag{5.41 b}$$

as the interval between A and B expressed as a number of equal mean solar days. This double formula replaces (5.35 a), which Ptolemy is unable to state without negative numbers (cf. page 141).

In the Almagest the theory of the equation of time is set forth in the briefest possible way. The procedure analysed above is condensed into a few phrases and there are no numerical examples to illustrate it; neither does Ptolemy tabulate $|E|$ as a function of $(\tau - \tau_0)$. However, such a table forms part of the later *Tabulae Manuales*.

What remains of the chapter is devoted to an estimate of how $|E|$ varies throughout the year. First Ptolemy calculates the maximum cumulative effect of the component

of E which depends on the inequality of the Sun's motion along the ecliptic. Next, he evaluates the maximum effect of the component depending on the obliquity of the ecliptic, and finally he adds the two effects. For the sake of brevity we shall not enter upon these considerations, the outcome of which is, of course, contained in the general formula (5.40).

CHAPTER 6

The Theories of the Moon

Introduction

Having finished his theory of the Sun, Ptolemy proceeds with the motion of the Moon. This disposition of the subject matter is not obvious although Ptolemy declares that he has adopted the logical order [IV, 1; Hei 1, 265]. From the pedagogical point of view it would have been more considerate towards the reader to go on with the relatively simple theories of the three superior planets Mars, Jupiter and Saturn instead of developing the much more intricate lunar theory. Actually, the reason for dealing with the Moon at this place becomes clear only later: the theories of the other planets are built on observations among which a certain number measure the distance between the planet and a fixed star. But the longitudes of the stars are determined by means of the position of the Moon (see page 240 ff.). Thus the ecliptic position of the planet cannot be found unless we know where the Moon is, i.e. unless we have a theory of its motion. This is another testimony to the fact that the Almagest is no student's text-book of astronomy, but a monograph on the construction of planetary theories.

The lunar theory of the Almagest is particularly important for the history of astronomy, and that for more reasons than one. First, the lunar theory has a special position in any theoretical system of astronomy since it must necessarily be geocentric, regardless of the geocentric or heliocentric character of the system in general. The lunar theories of the various systems are, therefore, particularly interesting for a comparison of the geometrical properties of such systems without respect to their physical interpretation. Second, the Moon is our nearest celestial body, and small irregularities of motion will be detected more easily in the Moon than in the planets, given a certain accuracy of observational methods. Thus it became one of Ptolemy's great contributions to the development of theoretical astronomy that he was able to discover a second anomaly of the Moon (see page 182) where Hipparchus had known only the first. Third, the geometrical model devised by Ptolemy in order to account for both these anomalies is in itself a testimony to the very high degree of ingenuity and sophistication with which he was able to cope with difficulties not even suspected by his Greek predecessors. Nowhere do we get a more convincing picture of Ptolemy as a great astronomer than here.

In the following we shall follow Ptolemy's construction of the lunar theory from the original Hipparchian hypothesis to the final model of his own. In the main outline of this exposition we are much indebted to a recent paper by Viggo M. Petersen (1969),

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History
of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_6,

who has made important contributions towards a better understanding of the logical steps of the construction of the model.

The Periods of the Moon's Motions

The theory of the Sun developed in Book III was a relatively simple affair. Seen from the Earth the motion of the Sun was circular and uneven, but it could be reduced to a uniform circular motion by means of the prosthaphairesis angle, which was a function of only one variable. In other words the Sun had one single anomaly only, which would be accounted for by the simple eccentric model.

In the Moon we have a much more complicated situation. Yet, at first sight, the motion of the Moon is similar to that of the Sun in so far as the Moon also proceeds steadily from West to East among the fixed stars, without any retrograde motion. Therefore we could develop a very simplified model assuming the Moon to move with a constant angular velocity upon a circle concentric with the ecliptic. Such a model would enable us to define the following characteristic periods:

1) The tropical month T_t as the period in which the Moon returns to the same point of the ecliptic, e.g. to the vernal equinox. This revolution implies an angular velocity ω_t determined by $T_t \cdot \omega_t = 360°$.
2) The sidereal month as the period of return to the same fixed star. This month is not used in the Almagest.
3) The synodic month T_s as the period between two successive conjunctions of the Sun and the Moon (i.e. two astronomical new Moons). The corresponding angular velocity ω_s determined by $T_s \cdot \omega_s = 360°$ measures the daily increase of the Moon's elongation. In a more general way the synodic month can be defined as the period of the lunations, i.e. the complete cycle of the Moon's phases.

This simple model breaks down for a number of reasons which are responsible for the fact that no one of the three periods defined here is constant. Thus we can only speak of T_t and T_s as the mean values of tropical and synodic months, just as the corresponding angular velocities ω_t and ω_s are mean daily motions.
4) Even crude observations are sufficient to show that the Moon has a very uneven motion in longitude, with an angular velocity varying from 10° to 14° per day. This means that we have to define a new period T_a called the mean anomalistic month in which the Moon returns to the same velocity. The corresponding angular velocity is found from $T_a \cdot \omega_a = 360°$.
5) Finally, the most obvious difference between the motions of the Sun and the Moon is that the latter does not follow the ecliptic, but has a varying latitude to the North or the South with a maximum of about 5°. Approximately the apparent path of the Moon can be described as a great circle intersecting the ecliptic at two diametrically opposed points called the ascending and descending nodes, and denoted in the following by ☊ and ☋ respectively. This gives rise to still another period, during which the Moon returns to the same latitude, e.g. to the same node. It is called the mean

draconitic month T_d and corresponds to an angular velocity determined by $T_d \cdot \omega_d = 360°$.

The Importance of Lunar Eclipses

In an introductory chapter [IV, 1; Hei 1, 365] Ptolemy deals with the question how the basic parameters of the lunar theory may be best determined from experience. Are there simple observations upon which a lunar theory can be founded, similar to the plain determination of the lengths of the seasons serving as basis for the theory of the Sun?

Here we must remember that the Moon is sufficiently near to the Earth to have a considerable daily parallax. The Ptolemaic value is 1°;7 [V, 13; Hei 1, 411] which compares well with the mean equatorial horizontal parallax of 57′ accepted to-day. This great value – almost twice the diameter of the Moon – means that most observations will be disturbed by parallactic effects, making it difficult both to compare observations made by observers at different points on the surface of the Earth, and to reduce them to an ideal observer at its centre as presupposed in a geocentric theory.

Only observations of lunar eclipses remain unaffected by parallax, since the passage of the Moon through the shadow of the Earth is an objective phenomenon and virtually the same for all possible observers. In this respect lunar eclipses are unlike eclipses of the Sun. Thus the most suitable observations to be used as data for a lunar theory are eclipses of the Moon. Since these take place at Full Moon, when the Moon and Sun are at diametrically opposed points of the heavens, an observation of lunar eclipse gives the geocentric ecliptic longitude of the Moon as 180° + the longitude of the Sun, the latter being found from the solar tables in Book III by means of the exact time of the eclipse. But it must be remembered that since these eclipses occur at Full Moons only, we have no guarantee that a theory built upon them will be able to account for other positions of the Moon.

However, not all lunar eclipses are equally suitable for the exact determination of the various periods. They must be carefully selected according to certain rules laid down in a not too clear chapter of the Almagest [IV, 2; Hei 1, 272] in which, as often elsewhere (see page 32), Ptolemy discusses the premises of a theory in terms of the theory itself.

Determination of the Periods

The determination of the lunar periods in the Almagest [IV, 2; Hei 1, 268 ff.] follows an old method based upon period relations, that is, the establishment of certain long intervals of time comprising whole numbers of the various periods. Thus Ptolemy ascribes to *the ancient mathematicians* a period S of $6585\frac{1}{3}$ days in which the Moon goes through 223 lunations, 239 revolutions in anomaly, 242 returns to the same latitude, and 241 revolutions + $10\frac{2}{3}$° in longitude relative to the fixed stars, in the same time as the Sun performs a little more than 18 sidereal revolutions. We have accordingly

$$S = 6585\tfrac{1}{3}^{d} = 18 \text{ sidereal years} + 10\tfrac{5}{6}^{d}$$
$$= 223\ T_s$$
$$= 239\ T_a$$
$$= 242\ T_d$$
$$\approx 241\ T_t\ (241 \text{ complete revolutions} + 10\tfrac{2}{3}^{\circ})$$

This is, in fact, the famous Saros period. The 'ancient mathematicians' referred to here were probably Babylonian astronomers who knew the Saros period and also used the sidereal year[1]).

The Greeks seem to have preferred the triple Saros, or Exeligmos[2]) period E containing

$$E = 19756^{d}$$
$$= 669\ T$$
$$= 717\ T_a$$
$$= 726\ T_d$$
$$= 723\ T_t\ (723 \text{ complete revolutions} + 32^{\circ})$$

Ptolemy states that Hipparchus found both the Saros and the Exeligmos periods too crude and replaced them by an even longer interval determined by comparing his own observations with Babylonian records. This Hipparchian period is

$$H = 126007^{d}1^{h} = 345 \text{ Egyptian years} + 82^{d}1^{h}$$
$$= 4267\ T_s$$
$$= 4573\ T_a$$
$$\approx 4612\ T_t\ (4612 \text{ complete revolutions} - 7\tfrac{1}{2}^{\circ})$$

From the period H we can find the length of the mean synodic month

$$T_s = \frac{126007^{d}1^{h}}{4267} = 29^{d};31,50,8,9$$

However, in the Almagest Ptolemy gives the values as

$$T_s = 29^{d};31,50,8,20$$

This is the only lunar period quoted in the Almagest [IV, 3; Hei 1, 278] which everywhere prefers to give the corresponding mean motions. The discrepancy is not great – about $\frac{1}{12}$ of a second – but it has not escaped the notice of astronomers, and already Copernicus commented upon it (*De Rev.* IV, 4, fol. 101v). It is most easily explained by the assumption that the value of T_s found in the Almagest was not derived from

1) The Saros period is attested by a procedure text from the city of Babylon found on two tablets (BM 55555 and BM 55562) identified and joined by A. Sachs in 1953. It is published in Neugebauer (ACT N° 210, pp. 271–273). The period is called *18 years of the Moon* and given as 1,49,45;19,20 days = 6585;19,20 days which Ptolemy rounds off to 6585;20 days. The Babylonian use of the sidereal year was already proved by Kugler (1900, p. 90 f.).
2) The Exeligmos period E is described by Geminus (*Astron.* XVIII, 3, p. 203 f.) according to whom it is due to Chaldaean astronomers. For the relative merits of the Saros and the Exeligmos periods as means of predicting eclipses, see Hartner 1969 a; cf. Huxley 1964.

the period H, but simply taken over from Babylonian astronomy where it is well attested (see Neugebauer, ACT, p. 78). Another discrepancy appears when we calculate the motion of the Sun in longitude during the period H by the daily mean motion given by (5.1). This gives

$$\begin{aligned} H &= 126007^{d}1^{h} \cdot 0°;59,8,17,13,12,31 \\ &= 356°;59,8 \\ &= 360° - 3°;0,52 \end{aligned}$$

or a lag of 3° in longitude instead of the $7\frac{1}{2}$° mentioned by Ptolemy. This looks like a computational error, but it can be explained also in another way. For if we divide H = 4267 synodic months by the Babylonian length of the year = 12;22,8 synodic months we find

$$\frac{4267}{12;22,8} = 344;58,42 \text{ years} = 345 - 0;1,17 \text{ years}$$

in which time the Sun performs 345 revolutions minus an arc of

$$360° \cdot 0;1,17 = 7°;42 \approx 7\tfrac{1}{2}°$$

in good agreement with what Hipparchus maintained (see Aaboe 1955). Another striking agreement between Hipparchus and his Babylonian predecessors appears if we, with Ptolemy [IV, 2; Hei 1, 272] divide H by 17. The resulting relation

$$H : 17 = 251\ T_s = 269\ T_a$$

is also well-known from astronomical cuneiform texts (see Neugebauer, ACT, p. 76). Finally we notice that the Hipparchian Period H contains no whole number of draconitic months. In order to determine T_d Hipparchus used another period relation, viz.

$$5458\ T_s = 5923\ T_d$$

which he is said to have found himself. In fact, this relation is another part of the Babylonian heritage as shown by Kugler (1900, p. 40).

These examples show that it is difficult to escape the conclusion that the period relations used by Hipparchus to establish the four fundamental periods of the motion of the Moon are of Babylonian origin[3]). The values of the corresponding mean angular

3) It is still an open question how the Babylonian astronomers made the observations which enabled them to establish the various period relations. The number of synodic months in a given interval of time can, of course, be found simply by counting lunations. The draconitic month may be determined from the intervals between eclipses. The anomalistic month presents a more difficult problem. Geminus (*Astron.* XVIII, 5, p. 203) thought that it was found from direct observation of the true daily motion of the Moon. This quantity is said to vary between a minimum lying between 11° and 12°, and a maximum between 15° and 16° (modern values are 11°;46 and 15°;21). These values would seem large enough to make it possible to find the date of, say, the minimal velocity with an error of one or two days even with crude observations of successive positions of the Moon relative to the fixed stars.

velocities (mean daily motions) found in the Almagest [IV, 3; Hei 1, 278 ff.] are the following, including some minor corrections mentioned below (page 180):

$$\begin{aligned}\omega_s &= 12°;11,26,41,20,17,59 \text{ per day}\\ \omega_a &= 13°;\ 3,53,56,17,51,59 \text{ per day}\\ \omega_d &= 13°;13,45,39,48,56,37 \text{ per day}\\ \omega_t &= 13°;10,34,58,33,30,30 \text{ per day}\end{aligned}$$

Compared with modern values these mean motions are astonishingly precise. In fact, ω_s is so accurate that the difference between the Ptolemaic and the modern value of the synodic month amounts to only a fraction of a second. On the other hand ω_t is influenced by the error of $\omega_\odot$ caused by the adopted value of the tropical year (see page 131). This causes the Moon to lose about 0°;25 in longitude per century (see Petersen, 1969, p. 149).

The Tables of Mean Motion

By means of these values Ptolemy constructs a set of *Tables of Mean Motions* of the Moon [IV, 4; Hei 1, 282] comprising mean motions in

$$\left.\begin{array}{l}\text{longitude}\\ \text{anomaly}\\ \text{latitude}\\ \text{elongation}\end{array}\right\}\ \text{with intervals of}\ \left\{\begin{array}{l}1^h\\ 1^d\\ 30^d\\ 1^a\ \text{(Egyptian)}\\ 18^a\ \text{(Egyptian)}\end{array}\right.$$

In the *Handy Tables* there is a similar set, except that the 18^a-period is replaced by a 25^a-period. In all such tables the term 'mean motion in longitude' is to be understood in the same sense as in the theory of the Sun (see page 133). Denoting the *mean longitude* of the Moon at the time t, reckoned from the vernal equinox, by $\lambda_m(t)$, we have for the mean daily motion in longitude

$$\frac{d\lambda_m(t)}{dt} = \omega_t \tag{6.1}$$

which can be integrated to give

$$\lambda_m(t) = \lambda_m(t_0) + \omega_t(t - t_0) \tag{6.2}$$

Here $\lambda_m(t_0)$ is the mean longitude at the epoch t_0. It is a constant of integration the value of which is determined later (see page 182). It is quoted at the head of the table of mean motion in longitude which only tabulates the term $\omega_t \cdot (t - t_0)$ as a function of the argument $(t - t_0)$. Latin astronomers called $\lambda_m(t_0)$ the *radix* of the *medius motus* in longitude (cf. page 153).

The other tables are constructed in a similar way. Thus the table of mean motion in anomaly lists $\omega_a \cdot (t - t_0)$, and the table of mean motion in latitude gives

$\omega_d \cdot (t - t_0)$. The expression is misleading since it does not refer to the variation of the latitude itself, but to the argument of the latitude function described later (page 201).

Finally the table of mean motion in elongation tabulates $\omega_8 \cdot (t - t_0)$. It is clear that relations anologous to (6.1) and (6.2) exist for all the tabulated functions, and that each table has its individual *radix*.

General Considerations

Besides the numerical values of the mean daily motions listed above, the lunar theory is founded upon two qualitative statements which follow from observations [IV, 2; Hei 1, 269]. They are

A The fact that the Moon can have its maximum (or mean, or minimum) velocity at any point on the ecliptic. This contrasts with the Sun, which always has its minimum velocity at the (supposedly fixed) apogee.

B The fact that the Moon can have its maximum northern (or southern) latitude at any point on the ecliptic. The same is, of course, true of the mean latitude $\beta = 0°$, which is seen from the fact that eclipses can occur everywhere on the ecliptic.

The problem is now how a theory accounting for the motion of the Moon can be constructed on this basis. In the case of the Sun the analogous problem was rather simple, there being only one main period (the tropical year) and one single anomaly in the sense that the Sun's deviation from a uniform circular motion could be described by a single correction $q(a_m)$ being a function of one variable only (see page 144).

But already at the beginning of the first theoretical chapter [IV, 5; Hei 1, 294] Ptolemy reveals that a similar simple model will prove insufficient in the case of the Moon, and that it will be necessary to introduce two independent 'anomalies' in order to account for the deviation from the uniform motion. The first (or great) anomaly is a deviation with a period of one anomalistic month. During a single revolution it depends only on the Moon's position on the ecliptic and is in some respects similar to the anomaly of the Sun, although there are certain differences (see A above). The second anomaly is a smaller deviation which depends on the relative position of the Moon and the Sun: thus it is a function of the elongation. It has a maximum at the two quadratures and is zero twice during one synodic month.

Ptolemy states that the second anomaly cannot be found until the first is known, while the first can be dealt with independently of the second. Ptolemy therefore proceeds in a pedagogical way with a model using one anomaly only. This First Lunar Model seems to derive from Hipparchus, who only knew the first anomaly of the Moon. Later Ptolemy modifies it in order to account for the second anomaly discovered by himself. In this way the final lunar theory is constructed step by step, a new modification being introduced everytime the previous model is contradicted by observations (cf. Petersen 1969).

We know from the solar theory that we have two geometrical tools for dealing with a motion with one anomaly, viz. 1) a model with an eccentric circle, and 2) a model with a concentric deferent and an epicycle. The two models were proved to be equivalent, describing the observations equally well, provided that the motion of the Sun on the epicycle, and the motion of the epicycle center had exactly the same period. This is not the case with the Moon, since the anomalistic month T_a is known to be greater than the tropical month T_t. The problem is then whether a similar equivalence can be established here. This is the subject matter of Book IV, Chapter 5, which investigates the two models in a general way without special reference to the Moon.

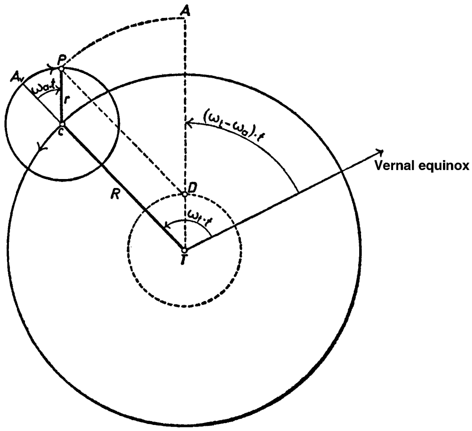


Fig. 6.1

Figure 6.1 shows a concentric deferent with its centre at T (the centre of the Earth), carrying the epicycle eastwards, at the same time as the epicycle carries the Moon P around C in the retrograde direction (as in the epicyclic theory of the Sun). This epicyclic model is thus characterized by the relation

$$\overrightarrow{TP} = \overrightarrow{TC} + \overrightarrow{CP} \tag{6.3}$$

where

$\overrightarrow{TC}$ is a vector with the constant length $|\overrightarrow{TC}| = R$ rotating around T with the constant angular velocity ω_t, and

$\overrightarrow{CP}$ is a vector with the constant length $|\overrightarrow{CP}| = r$ rotating around C with the constant angular velocity $\omega_a < \omega_t$. At the time $t = 0$ the two vectors are supposed to lie upon the same straight line in such a way that $|\overrightarrow{TP}| = R + r$. At the later time t the angles have the values indicated in the figure. It is easily seen that $\overrightarrow{TP}$ will perform a non-uniform rotation around T with the mean angular velocity ω_t.

We now construct the parallelogram TCPD in which the point D satisfies the relation

$$\overrightarrow{TP} = \overrightarrow{TD} + \overrightarrow{DP} \tag{6.4}$$

Here $\overrightarrow{DP}$ is equal and parallel to $\overrightarrow{TC}$, and $\overrightarrow{TD}$ is equal and parallel to $\overrightarrow{CP}$. Thus P can be conceived as lying upon a circle AP with centre D and radius DP = R. Therefore (6.4) defines an eccentric model for the motion of P equivalent to the epicyclic model defined by (6.3).

However – and this is the important difference from the solar theory – this equivalence depends on the fact that D is moving on a circle around T with the constant angular velocity $(\omega_t - \omega_a)$. The outcome of this investigation is, in other words, that it is possible to maintain the equivalence between an epicycle and an eccentric model if, and only if, the latter has a moving deferent with a centre D and an apogee A rotating around the centre of the Earth with constant angular velocity.

In the Moon this angular velocity is

$$\omega_t - \omega_a = 0°;6,41,2,15,38,31 \text{ per day} \tag{6.5}$$

and the corresponding period

$$T = \frac{360°}{\omega_t - \omega_a} \approx 8^a \cdot 85 \tag{6.6}$$

This makes the eccentric model much more complicated than that of the Sun. Partly for this reason – and partly in order to follow Hipparchus as far as possible – Ptolemy chooses to proceed with the epicyclic model with a fixed, concentric deferent [IV, 6; Hei 1, 300]. We shall later see how a moving deferent reappears in the second lunar model as a device for accounting for the second anomaly (page 186). It is worth noticing that already in the lunar theory [IV, 6; Hei 1, 306 f.] Ptolemy hints at the beautiful method of passing from an eccentric to an epicyclic model by means of an inversion on a fixed circle. We shall return to this method in connection with the theory of retrograde motions (page 334).

The First Lunar Model

The general features of the First Lunar Model can now be described briefly as follows [IV, 6; Hei 1, 301]: A circle is drawn in the plane of the ecliptic (see Figure 6.2), concentric with the latter, and situated inside the sphere of the Moon (this is one of the few places where the Almagest refers to the celestial spheres, cf. page 392). Another circle of the same size and also concentric with the ecliptic is situated in a plane inclined to the latter under an angle of 5° (the maximum latitude of the Moon). The common diameter of the two circles is the nodal line from ☊ to ☋; it rotates about the centre T from East to West with a daily motion of about 0°;3, drawing the inclined circle along; this device accounts for the fact B mentioned on page 165.

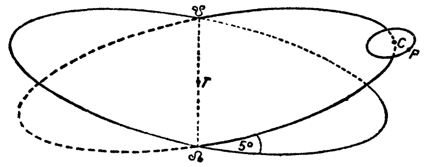


Fig. 6.2

The inclined circle is the proper lunar deferent upon which the centre C of the epicycle revolves from West to East with the mean angular velocity ω_t. The epicycle itself is a small circle in the inclined plane. It carries the Moon P which rotates from East to West (i.e. in the retrograde direction) with a mean angular velocity ω_a. Since ω_t and ω_a are different, the point of maximum apparent velocity will change from one lunation to another; cf. fact A, page 165.

It is easily seen that this model is able to simulate most of the observable lunar phenomena, at least in a qualitative way. Thus it will produce a direct motion in longitude (because of the motion of C) with varying velocity (because of the epicycle), but also a motion in latitude (because of the inclined plane) with a movable point of maximum latitude (because of the motion of the nodal line) and a movable point of maximum velocity (because $\omega_t \neq \omega_a$).

In the remaining part of Book IV Ptolemy investigates the problem whether the parameters of the model can be so defined that it also reproduces the phenomena in a quantitative way. But before entering upon numerical questions he makes the simplifying assumption that, as a first approximation, the motions in longitude and latitude can be treated independently [IV, 6; Hei 1, 302]. This means that the inclined plane is regarded as identical to the plane of the ecliptic as far as the calculation of longitudes is concerned. Furthermore he assumes that it is legitimate here to disregard the motion of the nodes. This gives the deferent a fixed position and a fixed origin in the plane of the ecliptic.

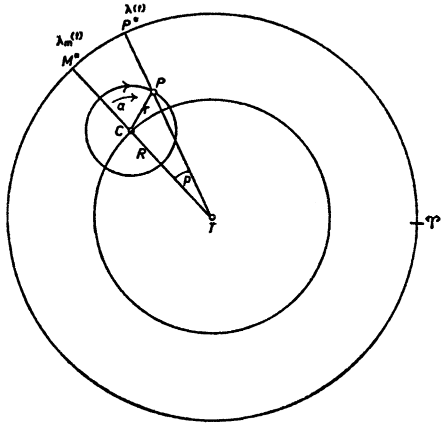


Fig. 6.3

By this reduction to the plane of the ecliptic the complete model is changed into the simplified plane model shown in Figure 6.3. Here the Moon P is seen at P* on the ecliptic with the true longitude $\lambda(t)$ from the vernal equinox. The vector TC points towards the point M* which might be called the ecliptic mean Moon. It moves from West to East with the constant angular velocity ω_t and is responsible for the direct motion of the Moon. Its longitude $\lambda_m(t)$ is given by (6.2).

The Moon itself rotates on the epicycle in the retrograde direction with the constant angular velocity ω_a. The angle M*CP = a(t) is an increasing, linear function of time called the anomaly, or, in Latin terminology the *argumentum* of the Moon. It is determined by the relation

$$a(t) = a(t_0) + \omega_a(t - t_0) \tag{6.7}$$

the last term of which is tabulated in the table of mean motion in anomaly. In the following a is called the argument.

The arc p = P*M* is equal to the prosthaphairesis angle PTC under which the epicycle radius CP is seen from the Earth. It satisfies the relation

$$\lambda(t) = \lambda_m(t) + p(a) \tag{6.8}$$

and can be expressed as a function of the single variable a (see page 179). This agrees with the fact that the First Model is constructed to account only for the first anomaly of the Moon's motion. It does so by the retrograde motion on the epicycle which causes the apparent velocity of the Moon to be greater when it is at the perigee of the epicycle than when it is at the apogee.

In order to complete the model we must now try to determine the numerical values of the parameters r, R, $\lambda_m(t_0)$ and $a(t_0)$.

The Empirical Basis of the Theory

The lunar theory in Book IV of the Almagest is founded upon a total of 15 eclipses, spread over almost 900 years, the first of which is an eclipse observed in Babylon in 721 B.C.: *According to the extant record it took place in the first year of Mardukempad on Thoth 29/30, was total, and began, it is said, a good hour after moonrise* [III, 6; Hei 1, 302]. They are all listed in chronological order in the following table. Here the first column contains a short symbol E_i for the eclipse in question. Column (2) is the date and place of the observation as given by Ptolemy, and column (3) is the time of the maximum, reduced to the longitude of Alexandria as explained below, and reckoned in equinoctial hours *from noon* (not from sunrise). In column (4) we have the corresponding longitude of the Moon when this is necessary. Column (5) gives the date of the eclipse according to Manitius, and column (6) gives a reference number to the general table of observations in Appendix A. Other eclipses are recorded and utilized in the theory of parallax (see page 208).

Lunar Eclipses in the Almagest

(1)	(2)	(3)	(4)	(5)	(6)
E_1	Mardukempad 1 Thoth 29/30 Babylon	$8^h;40$	$174°;30$	B.C. 721 March 19	1
E_2	Mardukempad 2 Thoth 18/19 Babylon	$11^h;10$	$163°;45$	B.C. 720 March 8	2
E_3	Mardukempad 2 Phamenoth 15/16 Babylon	$7^h;40$	$333°;15$	B.C. 720 Sept 1	3
E_4	Darius 20 Epiphi 28/29 Babylon	$10^h;45$		B.C. 502 Nov 19	6
E_5	Darius 31 Tybi 3/4 Babylon	$10^h;40$		B.C. 491 April 25	7
E_6	Nabonassar 366 Thoth 26/27 Babylon	$18^h;30$		B.C. 383 Dec 23	9
E_7	Nabonassar 366 Phamenoth 24/25 Babylon	$8^h;15$		B.C. 382 June 18	10
E_8	Nabonassar 367 Thoth 16/17 Babylon	$10^h;10$		B.C. 382 Dec 12	11
E_9	Nabonassar 546 Mesore 16 Alexandria	7^h		B.C. 201 Sept 22	30
E_{10}	Nabonassar 546 Mechir 9 Alexandria	$13^h;20$		B.C. 200 March 19	31
E_{11}	Calippus II, 55 Mesore 5 Alexandria	$14^h;15$		B.C. 200 Sept 12	32
E_{12}	Hadrian 9 Pachon 17/18 Alexandria	$8^h;24$		A.D. 125 April 5	54
E_{13}	Hadrian 17 Payni 20/21 Alexandria	$11^h;15$	$223°;15$	A.D. 133 May 6	63
E_{14}	Hadrian 19 Choiak 2/3 Alexandria	11^h	$25°;10$	A.D. 134 Oct 20	69
E_{15}	Hadrian 20 Pharmuti 19/20 Alexandria	16^h	$164°;5$	A.D. 136 March 6	73

Reduction of the Lunar Eclipses

How these original data are reduced to provide the Alexandrian time of the eclipse and the corresponding longitude of the centre of the Moon may be illustrated from the following scheme showing the reduction of E_1 [IV, 6; Hei 1, 302].

1	Character of eclipse	Total
2	Year of eclipse	Mardukempad 1
3	Date	Thoth 29
4	Hour	Began more than 1^h after moonrise
5	Approximate position of Sun	End of Pisces
6	Length of the day	12^h
7	Moonrise	6^h reckoned from noon
8	Beginning of eclipse	$7^h;30$ reckoned from noon
9	Duration of eclipse	4^h
10	Babylonian time of maximum	$9^h;30$ reckoned from noon
11	Alexandrian time of maximum t_1	$8^h;40$ reckoned from noon
12	True position of Sun	Pisces 24°;30
13	True longitude of Sun	354°;30
14	True longitude of Moon	174°;30

Here the first line indicates the magnitude of the eclipse which in this case is total, or equal to 12 digits, one digit being $\frac{1}{12}$ of the apparent diameter of the Moon [VI, 7; Hei 1, 500]. That the magnitude is e.g. 5 digits means that at the maximum or central phase $\frac{5}{12}$ of the diameter is inside the shadow of the Earth.

Lines 2–4 contain the date of the eclipse as found in the old records used by Ptolemy, and line 5 gives the approximate position of the Sun at this date. By means of this position we can find the length of daylight listed in line 6 by the method explained above (page 113). Since this length here happens to be 12^h the full Moon rises at 6^h after noon (line 7), so that the beginning of the eclipse can be fixed approximately at $7^h;30$ (line 8).

Now the duration of an eclipse depends on its magnitude. A total eclipse may last 4^h (line 9). The duration of apartial eclipse depends on the number of digits and can be found from a table given later [VI, 8; Hei 1, 519], or from experience. If half of the duration is added to the time of the beginning (line 8), the result is the time of the maximum at the place of observation (line 10). Now E_1 was observed in Babylon; but since it has to be referred to Alexandria we must subtract the difference in longitude between the two cities (50 minutes of time) in order to get the time t_1 when it was maximum in Alexandria (line 11).

By means of t_1 and the tables of the Sun we can now find the position of the Sun in signs and degrees to a better approximation than in line 5 (line 12). Line 13 gives this position as the solar longitude from the vernal equinox. Subtracting 180° we get the true ecliptic longitude $\lambda(t_1)$ of the Moon at the moment of centrality (line 14).

The Problem of Three Observations

The geometry of the First Model is defined by the length R of the deferent radius, and the radius r of the epicycle. Since Ptolemy is unable to determine the parallax of the Moon until he has a theory of its motion, he begins by giving R the value 60^p, where 1^p is an arbitrary unit of length – different, of course, from the unit employed in the solar theory. Then the problem is reduced to one of determining the value of r in the same unit. How this is done by means of 3 eclipse observations is the subject matter of the following section [IV, 6; Hei 1, 305].

Ptolemy chooses E_1, E_2, E_3, happening at the times t_1, t_2, t_3, when the Moon has the true longitudes λ_1, λ_2, λ_3 listed above, and occupies the positions P_1, P_2, P_3 on the epicycle; cf. the schematic arrangement in Figure 6.4.

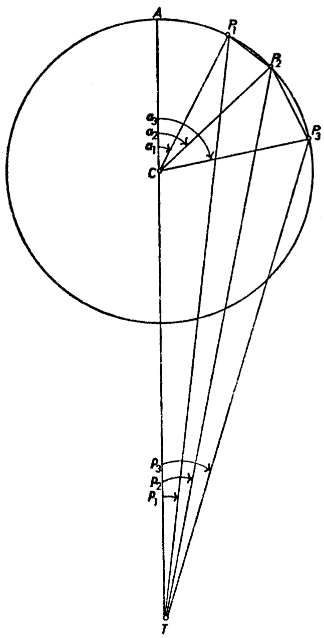


Fig. 6.4

Let us first consider the fact that the three longitudes λ_1, λ_2, λ_3 are different. This stems from two causes, the first of which is the motion of the epicycle centre C during the time intervals $(t_2 - t_1)$ and $(t_3 - t_2)$ between the eclipses, whereas the second is the motion of the Moon itself upon the epicycle during the same intervals. Together these two causes produce the observed displacements in longitude

$$\lambda_2 - \lambda_1 = 349°;15$$
$$\lambda_3 - \lambda_2 = 169°;30$$

The corresponding time intervals are (with regard to the equation of time)

$$t_2 - t_1 = 354^d 2^h 34^m$$
$$t_3 - t_2 = 176^d 20^h 12^m$$

The motion of the epicycle centre during these time intervals can be found from the table of the Moon's mean motion in longitude. We find

$$\lambda_{2m} - \lambda_{1m} = 345°;51$$
$$\lambda_{3m} - \lambda_{2m} = 170°;7$$

Subtracting we get

$$\begin{aligned}(\lambda_2 - \lambda_1) - (\lambda_{2m} - \lambda_{1m}) &= 3°;24 \\ (\lambda_3 - \lambda_2) - (\lambda_{3m} - \lambda_{2m}) &= -0°;37\end{aligned} \tag{6.9}$$

Since we have accounted for the motion of the epicycle centre, the two latter differences must be due to the changing position of the Moon upon the epicycle itself from P_1 to P_2, and from P_2 to P_3.

Further information on these three positions can be had from the table of the Moon's mean motion in anomaly during the two time intervals. Here we find

$$\begin{aligned}\alpha_1 &= a_2 - a_1 = 306°;25 = \text{arc } P_1P_2 \\ \alpha_2 &= a_3 - a_2 = 150°;26 = \text{arc } P_2P_3\end{aligned} \tag{6.10}$$

From the figure it is seen that the differences (6.9) are the angles under which the arcs P_1P_2 and P_2P_3 are seen from the centre T of the Earth, that is, they are the differences

$$\begin{aligned}\pi_1 &= p_2 - p_1 = 3°;24 \\ \pi_2 &= p_3 - p_2 = -0°;37\end{aligned} \tag{6.11}$$

between the prosthaphairesis-angles p_1, p_2, and p_3 under which the radius CP of the epicycle is seen from the Earth in the three successive positions P_1, P_2, and P_3. We can now state the problem to be solved in the following way:

> Find the radius r of a circle upon which two consecutive, given arcs $\alpha_1 = P_1P_2$ and $\alpha_2 = P_2P_3$ are seen under the given angles π_1 and π_2 from a given point T with the given distance R from the centre of the circle[4]).

Ptolemy's Solution

Ptolemy is aware that this problem is of a very general nature. We have already dealt with a similar case when he had to determine the eccentric orbit of the Sun from

4) This purely geometrical form of the problem is found as *Problema IV* in the second appendix to Vieta's *Apollonius Gallus*: *Dato circulo, & signatis in eius circumferentia tribus punctis, invenire punctum, a quo, cum ducentur tres lineæ rectæ ad signata puncta, inclinabuntur eæ ad angulos datos* (see *Francisci Vietæ Opera Mathematica*, rec. F. à Schooten, Leiden, 1646, p. 344). Vieta's solution was used in the solar theory by the Danish astronomer Willem Lange in his *De Annis Christi libri duo*, Leiden, 1649.

observations of the two equinoctial and one of the solstitial points. Later on we shall meet it again when we determine the eccentric equant circles of the planets (see page 273). In fact, it is a special case of the classical astronomical problem of deducing the elements of a planetary orbit from three observations. It must date back at least to Hipparchus[5]) since Ptolemy quotes his solution of it in the lunar theory (see page 146). But I think it is Ptolemy's own merit to have given so beautiful a solution in such general terms that it can be easily *applied to similar demonstrations regardless of whether – as here – we use an epicyclic or an eccentric hypothesis* [IV, 6; Hei 1, 306].

The solution is given in terms of a numerical example using the values (6.10) and (6.11) given above. Here we shall not follow the actual computation but illustrate the procedure in a more formalized way[6]) (see Figure 6.5).

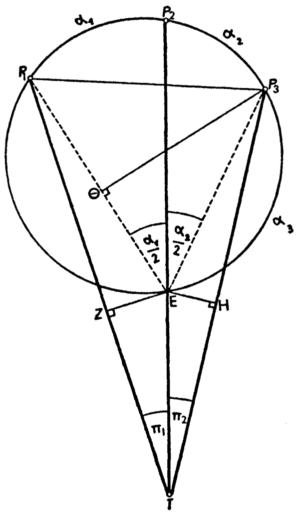


Fig. 6.5

First Ptolemy draws a straight line TP_2 from T to one of the three points (here P_2). It intersects the epicycle in E. Then EZ is drawn as a perpendicular to TP_1 and EH as a perpendicular to TP_3. Finally he draws the chords EP_1, EP_3, and P_1P_3.

5) As late as in the second edition of *The Exact Sciences in Antiquity* (1957, App. II, p. 210) O. Neugebauer adhered to the view that the method of determining the radius of the Lunar epicycle from three eclipses *was certainly used and probably invented by Hipparchus.* But in a later paper (1959, p. 14) he argued that the method may well be due to Apollonius; Tannery (1893, p. 206) was of the same opinion. The problem is too large to be dealt with here.
6) The following exposition is very close to that given by Neugebauer in *The Exact Sciences in Antiquity* (2nd edition, 1957, pp. 210–214). Also Delambre (II, 147 ff.) and Kempf (p. 11 ff.) follow Ptolemy rather closely. An interesting analytical treatment of the problem is found in Stumpff (I, pp. 30–33).

From the right angled triangles TEZ and P_1EZ Ptolemy deduces the relation

$$P_1E = d \cdot \frac{\sin \pi_1}{\sin\left(\frac{\alpha_1}{2} - \pi_1\right)} \tag{6.12}$$

where $d = TE$ is not yet determined; the expression follows directly from the sine relations applied to the triangle TEP_1. In a similar way we find

$$P_3E = d \cdot \frac{\sin \pi_2}{\sin\left(\frac{\alpha_2}{2} - \pi_2\right)} \tag{6.13}$$

From the triangle $P_3\Theta E$ we have

$$P_3\Theta = P_3E \cdot \sin \frac{\alpha_1 + \alpha_2}{2} \tag{6.14}$$

and

$$E\Theta = P_3E \cdot \cos \frac{\alpha_1 + \alpha_2}{2} \tag{6.15}$$

This enables us to find

$$P_1\Theta = EP_1 - E\Theta$$

and

$$P_1P_3^2 = P_1\Theta^2 + P_3\Theta^2 \tag{6.16}$$

But P_1P_3 can also be expressed by the law of chords as

$$P_1P_3 = 2r \cdot \sin \frac{\alpha_1 + \alpha_2}{2} \tag{6.17}$$

We can now write (6.16) in the form

$$\begin{aligned} &4r^2 \left(\sin \frac{\alpha_1 + \alpha_2}{2}\right)^2 \\ &= d^2 \left\{\left(\frac{\sin \pi_1}{\sin\left(\frac{\alpha_1}{2} - \pi_1\right)} - \frac{\sin \pi_2}{\sin\left(\frac{\alpha_2}{2} - \pi_2\right)} \cdot \cos \frac{\alpha_1 + \alpha_2}{2}\right)^2 + \right. \\ &\left. + \left(\frac{\sin \pi_2}{\sin\left(\frac{\alpha_2}{2} - \pi_2\right)} \cdot \sin \frac{\alpha_1 + \alpha_2}{2}\right)^2\right\} \end{aligned} \tag{6.18}$$

which determines the ratio

$$\frac{d}{r} = f(\alpha_1, \alpha_2, \pi_1, \pi_2) \tag{6.19}$$

as a function of the given angles.

To solve the problem we need another relation containing R, to obtain which we first find the arc $P_3E = \alpha_3$ by

$$P_3E = 2r \cdot \sin \frac{\alpha_3}{2} \tag{6.20}$$

which, combined with (6.13), gives

$$\sin \frac{\alpha_3}{2} = \frac{d}{r} \cdot \frac{\sin \pi_2}{2 \sin \left(\frac{\alpha_2}{2} - \pi_2\right)} \tag{6.21}$$

where d/r is known from (6.19). By means of α_3 we can then find the chord

$$P_2E = 2r \cdot \sin \frac{\alpha_2 + \alpha_3}{2} \tag{6.22}$$

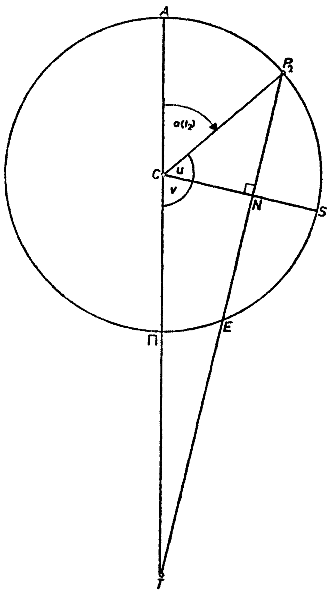


Fig. 6.6

In Figure 6.6, A is the apogee, Π the perigee, and C the centre of the epicycle. Applying Euclid II, 6 Ptolemy gets

$$TA \cdot T\Pi + C\Pi^2 = TC^2$$

where the first term is equal to $TE \cdot TP_2$. We have accordingly

$$d(d + P_2E) + r^2 = R^2$$

or

$$\left(\frac{R}{r}\right)^2 = 1 + \frac{d}{r}\left(\frac{d}{r} + \frac{P_2E}{r}\right) \tag{6.23}$$

Since d/r is known from (6.19) and P_2E/r from (6.22) we can now calculate R/r. Consequently we also have r/R, and the problem is solved.

The Parameters of the First Model

We can now determine the numerical values of the parameters. Here Ptolemy first puts $r = 60^p$. From (6.23) it follows that [IV, 6; Hei 1, 312]

$$R = 690^p;8,42$$

which again leads to

$$r = 5^p;13$$

if we use the standard length $R = 60^p$ of the radius of the deferent, as Ptolemy did in the solar theory (page 147) and as he does in the following theories of planetary motion; there is no evidence that Hipparchus used a similar normalisation of his geometrical models.

In order to check this value Ptolemy goes through exactly the same series of calculations as before, but now with his own eclipse triplet E_{13}, E_{14} and E_{15} as basis (see the table page 170). This time he finds the value $r = 5^p;14$. Since the two values are identical *without any perceptible difference* [IV, 6; Hei 1, 322; cf. Kempf 1878, p.32] Ptolemy finally accepts the round value

$$r = 5^p;15$$

as the size of the epicycle both in this and in the following versions of the lunar theory. He adds that this result is at variance with that of Hipparchus, who is said to have used first the triplet E_6, E_7, E_8, and second the triplet E_9, E_{10}, E_{11} as basis for his calculations. These were carried out according to both hypotheses (see page 166) with the result that Hipparchus found [IV, 6; Hei 1, 338]

$r = 6^p;15$ from the eccentric model, and
$r = 4^p;46$ from the epicyclic model

Hipparchus's calculation has been checked by Neugebauer (1959, p. 20) who found $r = 5^p;4$ instead of $4^p;46$, and $r = 6^p;26$ instead of $6^p;15$. Now, contrary to what many believe, says Ptolemy[7]), this discrepancy cannot be due to any difference be-

7) The two r-values found by Hipparchus present a difficult problem. Ptolemy's words *contrary to what many believe* cannot be taken as proof that it was Hipparchus who doubted the equivalence of the eccentric and epicyclic models. Neugebauer (1959, p. 18 ff.) suggests that Hipparchus may have suspected that the eccentricity of the lunar orbit (equivalent to the radius of the epicycle) did not remain constant – in other words that he may have had an inkling of a second anomaly not accounted for by the simple models. Now Ptolemy discovered a second anomaly of the Moon (see page 182); but this new irregularity does not appear at the syzygies. Therefore it became important for Ptolemy to show that the different Hipparchian r-values – derived from observations at the syzygies – were erroneous, "in order to protect his new theory against the argument that Hipparchus had found a variation of the eccentricity even at the syzygies" (ibid., p. 19). – Cf. also Toomer (1967).

tween the two models, the equivalence of which has been demonstrated beyond any doubt; but Ptolemy is able to clear up the matter, showing from a careful analysis of the data that Hipparchus not only worked from different triplets in the two cases, but also committed a number of errors when calculating the time intervals between the various eclipses [IV, 11; Hei 1, 344 and 347].

Having found the geometrical parameters of the model we can now determine the values of the angular parameters $a(t_2)$ and $\lambda_m(_2)$ with the time t_2 of the second eclipse as epoch (t_1 or t_3 would serve equally well). This is done by the following general method [IV, 6; Hei 1, 313]. In Figure 6.6 the line CNS is perpendicular to TP_2. In S this line bisects the arc P_2E which according to Figure 6.5 is equal to $(\alpha_2 + \alpha_3)$. For the argument (or 'anomaly') $a(t_2)$ we now have

$$a(t_2) = \text{angle } ACP_2 = 180° - u - v$$

where

$$u = \text{angle } P_2CS = \frac{\alpha_2 + \alpha_3}{2}$$

can be found from the relation

$$\left(\frac{R}{r}\right)^2 = 1 + \frac{d}{r}\left(\frac{d}{r} + 2\sin\frac{\alpha_2 + \alpha_3}{2}\right) \tag{6.24}$$

derived from (6.22) and (6.23), both d, R, and r being known. From the right-angled triangle TNC it follows that

$$\sin v = \frac{TN}{TC} = \frac{d + \frac{1}{2}P_2E}{R}$$

or, by (6.22)

$$\sin v = \frac{d}{R} + \frac{r}{R}\sin\frac{\alpha_2 + \alpha_3}{2} \tag{6.25}$$

Ptolemy finds in this particular case $u = 78°;35$ and $v = 89°;01$ with the result that

$$a(t_2) = 12°;24$$

The prosthaphairesis angle p_2 is the complement to v, or, with the proper sign

$$p_2 = -0°;59$$

Since the true longitude of the second eclipse was

$$\lambda(t_2) = 163°;45$$

we find from (6.8)

$$\lambda_m(t_2) = 164°;44 \tag{6.26}$$

As in the solar theory (page 151) Ptolemy prefers to reduce the angular parameters to the standard epoch t_0 of Nabonassar. To carry them back to this early date is

possible only if the mean motions are very exactly determined; consequently, Ptolemy first wants to verify or correct the mean motion values handed down to him from Hipparchian times. However, before this can be done he must consider the function p(a) defined by (6.8).

The Prosthaphairesis Function

Ptolemy declares very briefly [IV, 9; Hei 1, 335] that he has computed p(a) in the same way as in (the epicyclic version of) the theory of the Sun. This enables us to reconstruct

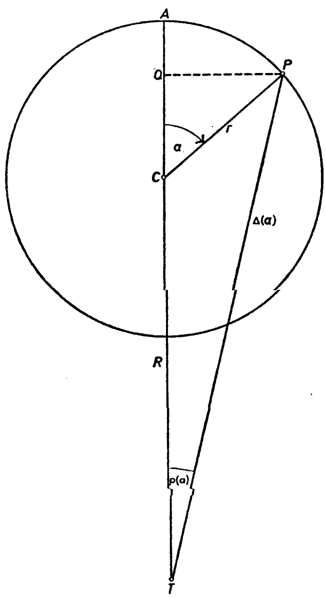


Fig. 6.7

his procedure by means of Figure 6.7 in which Q is the projection of P upon TA. We have here

$$QP = r \sin a$$
$$CQ = r \cos a$$

The distance TP = $\Delta(a)$ of the centre of the Moon from the centre of the Earth is

$$\Delta(a) = \sqrt{(r \sin a)^2 + (R + r \cos a)^2} \qquad (6.27)$$

and p(a) can be found from

$$\sin p(a) = \frac{-r \sin a}{\Delta(a)} \qquad (6.28)$$

or the equivalent formula

$$\tan p(a) = \frac{-r \sin a}{R + r \cos a} \tag{6.29}$$

where $R = 60^p$ and $r = 5^p;15$. This leads to the maximum value $p_{max} = 5°;1$. The resulting prosthaphairesis table, or *Table of Equations* [IV, 10; Hei 1, 337] is arranged in the same way as the table of the equation of centre in the solar theory with intervals of 6° between 0° and 90° and 3° from 90° to 180°. The symmetry relation

$$p(a) = -p(360° - a) \tag{6.30}$$

makes entries from 180° to 360° unnecessary.

Corrections of the Mean Motions

The first of these corrections [IV, 7; Hei 1, 324] is carried out comparing the eclipses E_2 and E_{14} which are separated by the time interval

$$t_{14} - t_2 = 854^a 73^d 23\tfrac{1}{3}^h = 311783^d 23\tfrac{1}{3}^h$$

which is much longer than the period H used by Hipparchus (see page 162) and thus leads to greater accuracy. For the mean motion in longitude we calculate from observations, using (6.8) and the table of equations,

$$\lambda_m(t_{14}) - \lambda_m(t_2) = 29°;30 - 164°;44 - 360° = 224°;46$$

The value of ω_t implied in Hipparchus' period relation is

$$\omega_t{}^H = 13°;10,34,58,33,30,30 \text{ per day}$$

and leads to

$$\lambda_m(t_{14}) - \lambda_m(t_2) = \omega_t{}^H \cdot (t_{14} - t_2) = 224°;46$$

This indicates that $\omega_t{}^H$ is in no need of correction.

In a similar way we find the mean motion in anomaly (or argument) from observation:

$$a(t_{14}) - a(t_2) = 64°;38 - 12°;24 = 52°;14$$

Here the period relation implies Hipparchus' value

$$\omega_a{}^H = 13°;3,53,56,29,38,38 \text{ per day}$$

which leads to the computed value

$$a(t_{14}) - a(t_2) = \omega_a{}^H \cdot (t_{14} - t_2) = 52°;31$$

Thus the Hipparchian value $\omega_a{}^H$ gives 0°;17 too much during the period $(t_{14} - t_2)$, or

$$\frac{0°;17}{t_{14} - t_2} = 0°;0,0,0,11,46,39 \text{ per day}$$

which subtracted from $\omega_a{}^H$ gives the corrected value

$$\omega_a = 13°;3,53,56,17,51,59 \text{ per day} \tag{6.31}$$

Finally Ptolemy wants to use the model for a better determination of the mean draconitic angular velocity, or the 'mean motion in latitude' ω_d [IV, 9; Hei 1, 326]. Here he considers the eclipses E_5 and E_{12} separated by the interval $t_{12} - t_5 = 615^a 133^d 21\frac{5}{6}^h$. The observations show that in both cases the Moon is eclipsed from the South to the amount of 2 digits, and that it is near the descending node. If we furthermore compute

$$a(t_5) = 100°;19 \quad \text{and} \quad a(t_{12}) = 251°;53$$

corresponding to the prosthaphairesis values

$$p_5 = -5°$$

and

$$p_{12} = 4°;53$$

we see that the two positions P_5 and P_{12} are fairly symmetrical on the epicycle and, therefore, equidistant from the Earth. All this means that between the two eclipses there is an integer number of 'revolutions in latitude', i.e. of periods T_d. But can that be verified?

From the period relation quoted on page 163 we find the Hipparchian value of

$$\omega_d{}^H = 13°;13,45,39,40,17,19 \text{ per day}$$

which gives the 'motion in latitude'

$$\omega_d{}^H \cdot (t_{12} - t_5) = 360° - 10°;2$$

But according to the model developed by Ptolemy, what is lacking in a complete number of revolutions can be found as

$$p_{12} - p_5 = 4°;53 - (-5°) = 9°;53$$

Accordingly there is a discrepancy amounting to

$$10°;2 - 9°;53 = 0°;9$$

so that it is necessary to correct $\omega_d{}^H$ by the small amount

$$\frac{0°;9}{t_{12} - t_5} = 0°;0,0,0,8,39,18 \text{ per day}$$

which added to $\omega_d{}^H$ gives the corrected value

$$\omega_d = 13°;13,45,39,48,56,37 \text{ per day} \tag{6.32}$$

It should be noticed that it was the corrected values of ω_a and ω_d which were quoted on page 164, and that the tables of mean motion were constructed from them [IV, 2; Hei 1, 277].

The Epoch of the Lunar Theory

So far we have used the time t_2 of the eclipse E_2 as a preliminary epoch of the lunar theory, but after the correction of the mean motions we can now reduce the parameters to the standard epoch t_0, i.e. the first year of Nabonassar, Thoth 1 at noon [IV, 8; Hei 1, 325]. For the time interval we have

$$t_2 - t_0 = 27^a 17^d 11\tfrac{1}{6}^h$$

We also had (6.26)

$$\lambda_m(t_2) = 164°;44$$

Computing $\omega_t(t_2 - t_0) = 123°;22$ we find immediately from (6.2)

$$\lambda_m(t_0) = 41°;22 \tag{6.33}$$

Similarly, from

$$a(t_2) = 12°;24$$

and

$$\omega_a(t_2 - t_0) = 103°;35$$

we find

$$a(t_0) = 268°;49 \tag{6.34}$$

by which values the parameters of the model are reduced to the standard epoch.

The Final Formula of the First Lunar Theory

Having found the values of the mean motions in longitude and anomaly at the standard epoch t_0 we can now write the final formula of the theory. Combining (6.2) and (6.8) we have

$$\lambda(t) = \lambda_m(t_0) + \omega_t \cdot (t - t_0) + p(a) \tag{6.35}$$

This formula enables us to determine the ecliptic longitude of the centre of the Moon at any given time t as the sum of the *radix* $\lambda_m(t_0)$, the tabulated mean *motus* $\omega_t(t - t_0)$, and the tabulated prosthaphairesis angle $p(a)$ corresponding to the mean argument at the time t. The latter is found by (6.7). This completes the first lunar theory. We see that the procedure for finding $\lambda(t)$ involves no more complicated calculations than addition and subtraction of sexagesimal numbers, found from tabulated values by means of simple linear interpolation. Actual trigonometrical computations are no longer necessary.

The Discovery of the Evection

As mentioned above (page 154) Ptolemy did not subject his theory of the motion of the Sun to any particular test, but in the lunar theory he has to be more circumspect.

So far this theory has been developed from observations of lunar eclipses, and the only test of it was that the model was able to describe other eclipses than those upon which it was founded [IV, 11; Hei 1, 339]. The result was that it could account for the longitude of the Moon at opposition (Full Moons) and, presumably, also at conjunctions (New Moons), i.e. at the two syzygies.

The problem is now whether the theory is able to give correct longitudes at other positions of the Moon relative to the Sun, e.g. at the quadratures. In order to answer this it is necessary to know such positions from experience, and this implies observations of another kind than the simple determination of the time of an eclipse. Therefore Ptolemy begins the 5th Book of the Almagest with a description of a new kind of astronomical instrument designed to that purpose. It is called an *astrolabon*[8]), but has nothing to do with the instrument later called an astrolabe. It is rather an armillary sphere, consisting of a number of graduated rings representing the fundamental circles of the heavens. When a diopter is adjusted to a star its longitude and latitude can be read directly off the instrument [V, 1; Hei 1, 350].

The first observation of the new kind recorded in the Almagest [V, 3; Hei 1, 362] was made by Ptolemy at a time t defined as the 2nd year of Antoninus, $5\frac{1}{4}$ hour before noon on the 25th of Phamenoth (Appendix A, N° 82). It gave as result the

$$\begin{array}{ll} \text{longitude of the Sun} & = 318°;50 \\ \text{longitude of the Moon} & = \underline{219°;40} \\ \text{and therefore the elongation} & 99°;10 \end{array}$$

This means that the Moon is a little more than 90° behind the Sun, i.e. it is approaching its third quarter. We must now compute the position of the Moon at the given time t in order to see if the theory agrees with this observation.

Ptolemy begins by checking the position of the Sun by means of the solar tables according to the formula (5.33), where

$$t - t_0 = 885^a 203^d 18\tfrac{3}{4}^h$$

The longitude of the Sun is found to be $\lambda(t) = 318°;50$ in agreement with the observed value (it seems that the correctly computed value is 318°;43).

In the same way Ptolemy computes the mean longitude of the Moon (i.e. the longitude of the epicycle centre). The result is

$$\lambda_m(t) = 227°;20$$

which compared with the observed longitude above

$$\lambda(t) = 219°;40$$

shows a difference of 7°;40. In other words, the true Moon is 7°;40 behind the mean Moon.

8) Concerning the *astrolabon*, see the reconstruction (by P. Rome) published by A. Rome (1927, cf. Theon I, p. 5) and based on Pappus' description. The instrument was also described by Proclus (*Hyp.* VI, 1–25). – Cf. Dicks, 1954, p. 81 ff.

Now the difference $\lambda - \lambda_m = p$ is the prosthaphairesis angle which according to (6.28) or to the table [IV, 10; Hei 1, 337] has a maximum value of 5°;1. This means that the model is unable to account for all of the observed difference. How great the prosthaphairesis deduced from the model is at the time of observation is easily found if we compute the argument a(t) by means of the lunar tables according to the formula (6.7). This gives a(t) = 87°;19 from which follows a theoretical prosthaphairesis equal to p(a) = 4°;58. In this case the model is, therefore, unable to account for 2°;42 of the observed difference $(\lambda - \lambda_m)$.

Several other observations, including one made by Hipparchus (Appendix A, N° 48), corroborate the fact that the total observed prosthaphairesis can amount to 7°;40 at quadratures. This means that whereas the theory satisfies the observations at the syzygies (where p_{max} = 5°;1) it fails at the quadratures (where p_{max} = 7°;40). This cannot be explained on the assumption that the formula (6.28) for p(a) is wrong. This is clear from the facts that (1) this formula holds good at the syzygies, and (2) the excessive amount of prosthaphairesis is a function of the elongation and thus correlated to the motion of the Sun, but – unlike (6.28) – independent of the motion of the Moon upon the epicycle described by the argument a(t).

The only way out is to acknowledge that besides the first anomaly described by (6.28) the Moon must have an independent second anomaly. The discovery of the latter – later called the evection – is one of Ptolemy's greatest personal contributions to astronomy.

The Second Lunar Model

The problem of accounting for the evection is solved by Ptolemy in a masterly way, but at the cost of substituting for the First (Hipparchian) Model a much more sophisticated Second Model which must accordingly be Ptolemy's own invention. This implies two major changes.

First, the evection means that the apparent size of the epicycle is larger at the quadratures than at the syzygies. Accordingly we must supply the model with a device drawing the epicycle centre C nearer to the Earth at the former points than the 60^p which is its distance at the latter. This is achieved by replacing the concentric deferent of the First Model by an eccentric deferent.

Second, the evection is a function of the elongation of the Moon from the Sun. Consequently the Second Model must be provided with a mechanism coupling the motion of the epicycle centre to the motion of the Sun. This is done by letting the centre of the eccentric deferent rotate upon a small circle around the centre of the Earth, a device which we have met with already in the general theoretical considerations above (page 167).

The eccentricity e = TD of the new eccentric deferent can be found from Figure 6.8 in which D is the centre of the deferent, A its apogee and Π its perigee [V, 4; Hei 1, 365]. The epicycle is supposed to have the same radius $r = 5^p;15$ as in the First Model.

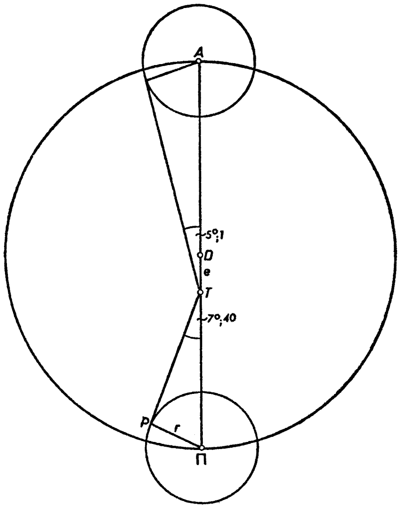


Fig. 6.8

At the apogee it is seen with its normal size, which implies that we must put TA $= 60^p$. At the perigee the epicycle radius is seen under the angle 7°;40. We have accordingly

$$T\Pi = \frac{5^p;15}{\sin 7°;40} = 39^p;22$$

The diameter of the eccentric deferent is, therefore

$$A\Pi = TA + T\Pi = 99^p;22$$

and its radius

$$R = 49^p;41 \tag{6.36}$$

Finally, the eccentricity will be

$$e = TD = 60^p - 49^p;41 = 10^p;19 \tag{6.37}$$

measured in such unit in which $R = 49^p;41$. If we want to give the deferent the standard radius 60^p we must change the eccentricity in the ratio

$$\frac{e}{60^p} = \frac{10;19}{49;41}$$

which gives

$$e = 12^p;28 \tag{6.38}$$

We saw that the evection was maximum at both quadratures. Consequently the epicycle centre C must be at the perigee Π of the deferent at both the first and third

quarter. It follows that it must be at the apogee A at both syzygies. In other words, C performs two revolutions around the deferent in the course of one synodic month. But since C is at A at each conjunction (as well as at each opposition) it follows that A must move in such a way that it follows the place of the Sun from one conjunction to the next following. This is the reason for the motion of the apogee of the deferent which, of course, implies a similar motion of its centre D.

In this way emerges the Second Model [V, 2; Hei 1, 355] which is illustrated in

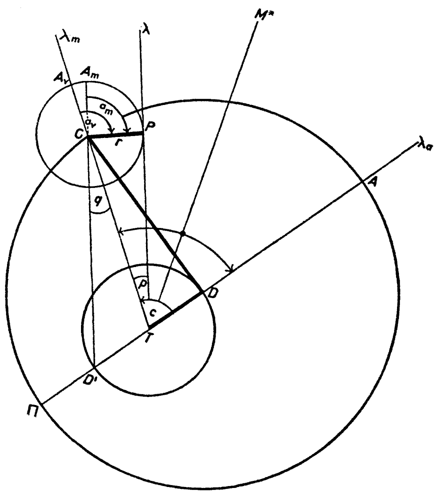


Fig. 6.9

Figure 6.9. The position P of the Moon is determined by the vector

$$\overrightarrow{TP} = \overrightarrow{TD} + \overrightarrow{DC} + \overrightarrow{CP} \tag{6.39}$$

Here $\overrightarrow{TD}$ is the eccentricity vector with a length of $e = 10^p;19$. It rotates from East to West with the constant angular velocity ω_S relative to the line TM* from the Earth to the ecliptic mean Sun M*. Consequently the deferent centre rotates on a small circle with the centre T and the radius $e = 10^p;19$, the *circulus parvus* of the Latin astronomers.

The deferent vector is $\overrightarrow{DC}$. It has a length of $R = 49^p;41$ and rotates about D with a non-uniform angular velocity determined by the fact that the line TC rotates about T with the constant angular velocity ω_S relative to TM*, and in the direction from West to East. It follows that C moves upon a circle (the deferent) with an angular velocity which is non-uniform with respect to the centre D of this circle, but uniform with respect to another point T. This is the first instance where Ptolemy violates the principle of uniform, circular motion in the strict sense (see page 35) without even

mentioning this departure from a fundamental dogma in the philosophy of astronomy.

Finally the epicycle vector $\overrightarrow{CP}$ has a length of $r = 5^p;15$. It rotates about C with a constant angular velocity ω_a relative to the line TCA_v from the Earth through C to the apogee A_v of the epicycle.

The Behaviour of the Second Model

In Figure 6.9 the line TM* to the mean Sun rotates from West to East with the constant angular velocity $\omega_{\odot}$ given by (5.1). Thus the angle M*TC is the mean elongation of the Moon, i.e. the angular distance between the ecliptical mean Moon and mean Sun. Since the lines TA and TC move away from TM* with the same angular velocity but in opposite direction, it follows that at any time the angle ATC is equal to twice the elongation. Mediaeval Latin astronomers gave this angle the convenient name of the *centrum lunae* since it measures the angular distance of the epicycle centre from the (moving) apogee of the deferent.

We can now see that in the course of one synodic month T_s, beginning at $t = 0$ with the epicycle centre at A, the following phenomena are produced by the model:

$t = 0$ Both C and A are at M*. The elongation is 0° and Sun and Moon are in mean conjunction (mean New Moon).

$TC = 60^p$ and $p_{max} = 5°;1$

$t = \frac{1}{4} \cdot T_s$ C has moved through half of the deferent to the perigee Π opposite to A. The mean Sun is in the middle between A and Π; the elongation is 90° and the Moon is in a mean quadrature (1st quarter) with the Sun.

$TC = 39^p;22$ and $p_{max} = 7°;40$

$t = \frac{1}{2} \cdot T_s$ C has moved through another half of the deferent and is back at A while Π is now at M*. The elongation is 180°, and Sun and Moon are in mean opposition (mean Full Moon).

$TC = 60^p$ and $p_{max} = 5°;1$

$t = \frac{3}{4} \cdot T_s$ C has now reached Π once more and M* is in the middle between Π and A. The elongation is 270°, and the Moon is in a mean quadrature (3rd quarter) with the Sun.

$TC = 39^p;22$ and $p_{max} = 7°;40$

$t = T_s$ Having completed two revolutions on the deferent C again returns to M*, meeting A there, and restoring the situation from $t = 0$.

The motions of the various vectors of the model are listed here:

Daily Motion of	Relative to	Direction	Numerical Value
TM*	T♈	East	$\omega_{\odot} = 0°;59,\ 8$
TC	TM*	East	$\omega_s = 12°;11,26$
TC	T♈	East	$\omega_t = 13°;10,34 = \omega_{\odot} + \omega_s$
TA	TM*	West	$\omega_s = 12°;11,26$
TA	TC	West	$2 \cdot \omega_s = 24°;22,52$
TA	T♈	West	$\omega_s - \omega_{\odot} = 11°;12,18$
CP	CA_v	West	$\omega_a = 13°;\ 3,53$
TC	T☊	East	$\omega_d = 13°;13,45$
TA	T☊	West	$2\omega_s - \omega_d = 11°;\ 9,\ 7$

The two last angular velocities of TC and TA relative to the ascending node are unimportant to the theory of longitudes, but essential to the computation of latitudes.

These parameters complete the main features of Model II. It is characteristic of Ptolemy's manner of exposition that he explains the geometrical and kinematical structure of the new model almost at the beginning [V, 2; Hei 1, 354] and before explaining the observation which forced him to abandon the First Model [V, 3; Hei 1, 361] (cf. page 32).

Ptolemy always conceives the motion of the epicycle centre C as a circular motion around the (moving) centre D of the deferent. He never asks for the orbit described by C relative to the centre T of the Earth. It can be determined in the following way. Suppose that at $t = 0$ we have a mean conjunction upon the line from T to the vernal equinox (cf. Figure 6.10). Let this line be the polar axis and T the origin of a polar system of co-ordinates. At the time t the vector TC forms an angle

$$\varphi = \omega_t \cdot t$$

with the axis. At the same time TC forms an angle $= 2\omega_s \cdot t$ with TD. From the triangle DTC we have

$$R^2 = e^2 + \rho^2 - 2e\rho \cos (2\omega_s \cdot t)$$

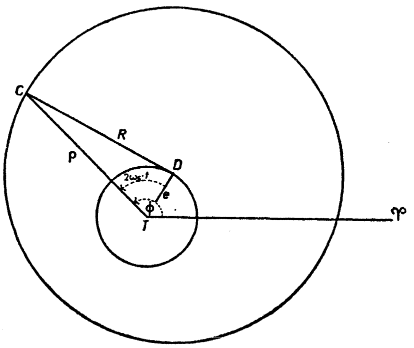


Fig. 6.10

where $\rho = |TC|$. Solving this equation and eliminating t we find

$$\rho = e \cos\left(\frac{2\omega_s}{\omega_t} \cdot \varphi\right) \overset{+}{\underset{(-)}{}} \sqrt{R^2 - e^2 \cdot \sin^2\left(\frac{2\omega_s}{\omega_t} \cdot \varphi\right)} \qquad (6.40)$$

as the equation of the locus of C. It is seen that $\rho(\varphi)$ has a maximum $\rho_{max} = R + e$ for

$$\frac{2\omega_s}{\omega_t} \cdot \varphi = p \cdot 360°$$

where p is an integer, i.e. for

$$\varphi = p \cdot 180° \cdot \frac{\omega_t}{\omega_s}$$

or, approximately

$$\varphi = p \cdot 195°$$

Similarly there will be a minimum $\rho_{min} = R - e$ for

$$\varphi = 90° + p \cdot 195°$$

The general form of the curve is shown in Figure 6.11.

A Test of the Second Model

Before making the Second Model fit for practical calculations (by computing a prosthaphairesis table) Ptolemy wants to subject it to a critical test. As we have seen, the model was constructed to give correct positions both at the syzygies and the quadratures. The problem is whether it is able to account for observations at other points as well. Here Ptolemy draws attention [V, 5; Hei 1, 367] to a 'peculiarity' occurring near the points where the Moon is sickle-shaped, i.e. near the octants. Here the Moon deviates a little from the theoretical position derived from Model II, so that this model has to be changed once more into the still more sophisticated Model III. As before, Ptolemy describes the main features of the new model before telling us of the observations forcing him to introduce it. Here we shall proceed in the opposite way.

One of these octant-observations (Appendix A, N° 49) was made by Hipparchus at Rhodes at a time t given as the year 197 after the death of Alexander, on Phamenoth 11, when at the beginning of the second hour the Sun had the ecliptic longitude 37°;45. The observed longitude of the Moon was 351°;40 which corrected for parallax gave the true, geocentric longitude of 351°;27,30. It follows that the distance between the true Sun and the true Moon was 313°;42,30, i.e. the Moon was near the 7th octant.

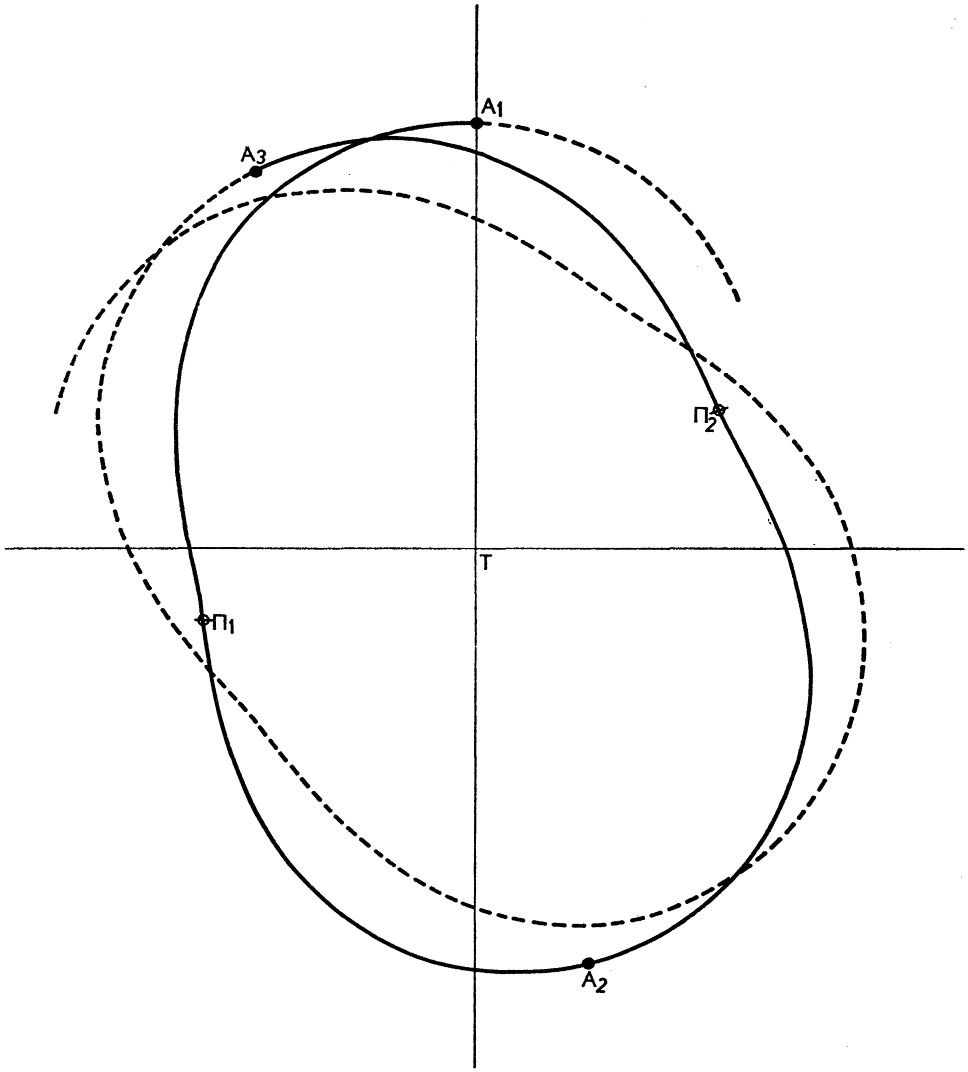


Fig. 6.11

The time difference

$$t - t_0 = 620^a 219^d 18^h$$

enables us to compute the longitude of the Moon from the tables of mean motion. Ptolemy finds

1) mean longitude of the Sun = 36°;41
2) true longitude of the Sun = 37°;45 as observed
3) mean longitude of the Moon = 352°;13
4) argument of the Moon = 185°;30

From 2) and 3) follows a distance between the true Sun and the mean Moon of 314°;28, differing from the observed distance between the true Sun and the true Moon by an amount of 0°;46 which obviously is the prosthaphairesis angle of the Moon at the time of observation.

The problem is now whether this prosthaphairesis angle is compatible with the computed value 185°;30 of the argument, i.e. the angular distance A_vP between the Moon P and the true apogee A_v of the epicycle (the point of the latter furthest away from the Earth). This cannot be answered immediately since a prosthaphairesis table for Model II has not yet been constructed. We must therefore solve the problem by direct computation.

The following figure 6.12 shows the configuration of Model II at the time of the observation [V, 5; Hei 1, 370]. The outer circle represents the ecliptic upon which are shown the positions of the true Sun S*, the Mean Sun M*, the mean Moon C* and the true Moon P*, the latter being 0°;46 behind C*. The angle C*TM* is seen to be 44°;28, which according to Model II (see page 187) is equal to the angle ATM*.

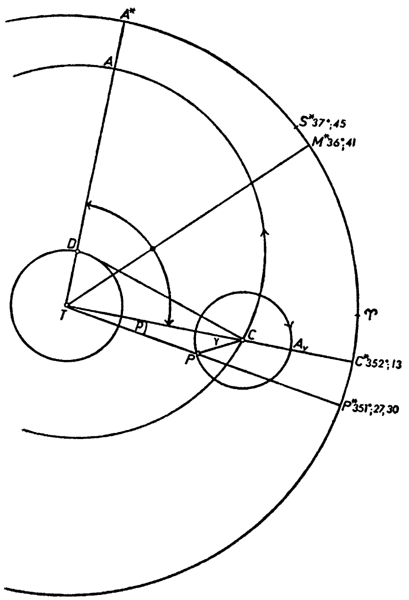


Fig. 6.12

Twice this value, or 88p;56, is equal to the angle DTC. In triangle DTC we have TD = 10p;19 and DC = 49p;41. By means of a trigonometrical computation we can then find the distance of the epicycle centre from the centre of the Earth TC = 48p;48.

Let us now consider triangle TCP in which we have the prosthaphairesis angle

CTP = 0°;46. Since CP = 5^p;15 and TC = 48^p;48 we can compute the angle TCP = γ by means of

$$\frac{\sin(0°;46 + \gamma)}{48^p;48} = \frac{\sin 0°;46}{5^p;15}$$

which according to Ptolemy gives the value γ = 6°;21. Thus the argument corresponding to the given prosthaphairesis angle of 0°;46 would be 180° − 6°;21 = 173°;39 instead of the value 185°;30 derived above from the table of mean motion in anomaly. This discrepancy is precisely the 'peculiarity' occurring at the octants. Actually, the maximum effect of the discrepancy occurs at elongations of 57° and 123° (see Kempf (quoting Chasles) 1878, p. 25).

The Third Lunar Model

Perhaps one would think that in order to account for this discrepancy at the octants Ptolemy had now to introduce a third 'anomaly' of the Moon, just as he had to introduce the second anomaly in order to account for the observed deviations at the quadratures (see page 184). But this would have been unnecessary and misleading. Since the new discrepancy is found at the octants, it is dependent on the relative position of the Sun and Moon just as the second anomaly. It belongs to the same order of anomalies as the latter and must be accounted for with as small changes of Model II as possible.

Ptolemy tries to account for the new phenomenon by giving the epicycle what he calls a certain πρόσνευσις or inclination. In fact he assumes that the Moon really has an argument of 185°;30, but that this angle must be reckoned from a point A_m, called the m e a n a p o g e e of the epicycle and different from its t r u e a p o g e e A_v. It follows from what we found above that the mean apogee must be situated 185°;30 East of the observed position of the Moon, in other words it must be placed 185°;30 + 6°;21 = 191°;51 East of the true perigee, i.e. 11°;51 East of the true apogee A_v. The situation is now as shown in Figure 6.13, from which a curious geo-

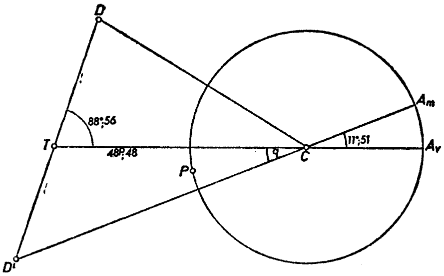


Fig. 6.13

metrical construction of the mean apogee A_m is deduced. The line A_mC prolonged beyond the epicycle centre C cuts the apogee line DT of the deferent in the point

D′ which we must now determine. In triangle TCD′ we have TC $= 48^p;48$ and all the angles are known. It is therefore easy to compute

$$TD' = 10^p;17$$

This is very nearly equal to the radius TD $= 10^p;19$ of the small concentric circle upon which the deferent centre D revolves. Therefore Ptolemy does not hesitate to define D′ as the point on the concentric circle diametrically opposed to the deferent centre D.

The introduction of the *prosneusis*, resulting in the replacement of the true apogee A_v of the epicycle by the mean apogee A_m as the point from which the argument is reckoned is the only difference between Model II and Model III. The latter is shown in Figure 6.14. It remains the final model of Ptolemy's lunar theory, and he does not

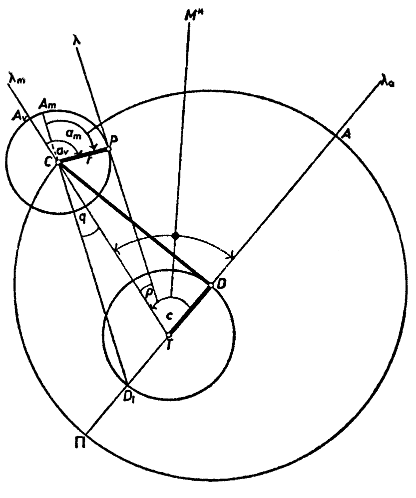


Fig. 6.14

try to refine it further by means of more observations outside the octant points. This means that most of the variables are unaffected by the transition. Thus the mean longitude $\lambda_m(t)$, the true longitude $\lambda(t)$, and the **centrum**, or **double elongation**

$$c(t) = 2 \cdot \{\lambda_m(\text{Moon}) - \lambda_m(\text{Sun})\} \tag{6.41}$$

are defined precisely as before.

Only the variable governing the motion on the epicycle is influenced by the change. Since we now have a mean and a true apogee, we must distinguish between a

$$\text{mean argument } a_m(t) = \text{angle } A_mCP \tag{6.42}$$

and a

$$\text{true argument } a_v(t) = \text{angle } A_vCP \tag{6.43}$$

The former is the variable called the *argumentum medium* by Mediaeval Latin astronomers. It measures the distance of P from the mean apogee A_m (the *aux media*) and is the same function as (6.7). However, the latter relation must now be written

$$a_m(t) = a_m(t_0) + \omega_a(t - t_0) \tag{6.44}$$

On the other hand the true argument a_v (the *argumentum verum*) measures the distance of P from the true apogee A_v (the *aux vera*). It is a non-linear function of time, connected with $a_m(t)$ by the relation

$$a_v(t) = a_m(t) + q(c) \tag{6.45}$$

where q(c) is equal to the arc A_vA_m, that is to the angle under which the radius TD′ of the small, concentric circle is seen from the epicycle centre C. It is called the *prosthaphairesis of anomaly*, or, in Latin terminology, the *equatio centri* (equation of centre).

Considering the triangle DTC, we have for the distance TC = ρ(c) of the epicycle centre from the Earth

$$R^2 = e^2 + \rho^2 - 2e\rho \cos c$$

which is solved by

$$\rho(c) = \sqrt{R^2 - e^2 \cdot \sin^2 c} + e \cos c \tag{6.46}$$

(cf. 6.40). From the triangle TD′C we then have

$$\tan q(c) = -\frac{e \sin c}{\rho(c) + e \cos c} \tag{6.47}$$

from which the equation of centre can be determined as a function of the single variable c(t). Ptolemy gives a numerical example [V, 6; Hei 1, 380] equivalent to the formula

$$\sin q(c) = -\frac{e \cdot \sin c}{\sqrt{(e \cdot \sin c)^2 + (\rho(c) + e \cos c)^2}} \tag{6.48}$$

which follows from (6.47) by a simple trigonometrical relation. He tabulates q(c) in column 3 of a general table of equations [V, 8; Hei 1, 390] which will be described in more detail below (page 197).

Let us finally consider the angle CTP = p under which the epicycle radius CP is seen from the centre of the Earth. It is called the *prosthaphairesis of longitude* or in Latin the *aequatio argumenti*, that is, the equation of argument. It is seen that the fundamental relation (6.35) is still true, but that p is no longer the simple function (6.29) of the argument only. Ptolemy gives a numerical example [V, 6; Hei 1, 382] of how it can be computed for particular values of the centrum c and the true argument

a_v. However, it is obvious that we can apply the procedure leading to (6.29); we only have to substitute in (6.29) the true argument a_v for a, and the distance TC $= \rho(c)$ given by (6.46) in order to express $p(c, a_v)$ as a function

$$\tan p(c, a_v) = -\frac{r \sin a_v}{\rho(c) + r \cos a_v} \tag{6.49}$$

of the two variables $c(t)$ and a_v. Also here Ptolemy computes p by a procedure corresponding to the formula

$$\sin p(c, a_v) = -\frac{r \sin a_v}{\Delta(c, a_v)} \tag{6.50}$$

which is equivalent to (6.49) since

$$\Delta(c, a_v) = \sqrt{(r \sin a_v)^2 + (\rho(c) + r \cos a_v)^2} \tag{6.51}$$

is the distance TP between the centres of the Earth and the Moon, cf. the corresponding formula (6.27) for the First and Second Models.

Programme for the Calculation of Longitudes

The general procedure for finding the true longitude of the centre of the Moon at a given time t can now be traced in the following steps:

1° Find the time interval $(t - t_0)$ between t and t_0 expressed in astronomical units (i.e. taking the equation of time into account if necessary, cf. page 157).
2° Find $\lambda_m(t)$ from the table of mean motions in longitude, cf. (6.2).
3° Find $a_m(t)$ from the table of mean motions in anomaly, cf. (6.44).
4° Find the mean longitude of the Sun at the time t by means of the solar theory (cf. page 133).
5° Find the centrum $c(t)$ as twice the mean elongation from 2° and 4°, cf. (6.41).
6° Find the distance $\rho(c)$ given by (6.46).
7° Find the equation of centre $q(c)$ from col. 3 of the general table of equations, cf. (6.47).
8° Find the true argument a_v by (6.45) and the values found in 3° and 7°.
9° Find the equation of argument $p(c, a_v)$ by (6.49), using the values found in 5°, 6°, and 8°.
10° Finally, find $\lambda(t)$ by (6.8) using the values found in 2° and 9°.

It appears that steps Nos 6° and 9° involve rather complicated trigonometrical calculations. This Ptolemy tries to avoid through the construction of a table from which the function $p(c, a_v)$ can be determined by simple arithmetical operations.

Approximation for the Prosthaphairesis Function

The function $p(c, a_v)$ given by (6.49) depends on the two variables c and a_v. Thus it could be tabulated in a table with double entry. The symmetrical properties of (6.49) would make it possible here to use arguments in the intervals of $0° < c < 180°$ and $0° < a_v < 180°$ only. But even if we restricted ourselves to intervals of 3° of both variables the result would be a table of 61 rows, 61 columns, and 3721 tabulated values. The enormous work of calculating such a table is avoided by Ptolemy by an ingenious method of approximation, based on the fact that the true argument a_v is seen to be a 'strong' but the centrum c a 'weak' variable in the sense defined above (page 85). The result is a procedure [V, 6–7; Hei 1, 380] which can be summarized in the following formula (cf. 3.57)

$$p(c, a_v) = p_1(a_v) + \{p_2(a_v) - p_1(a_v)\}\frac{f(c)}{60} \tag{6.52}$$

the various terms of which are given as functions of one variable only in the columns 4, 5, and 6 in the general table [V, 8; Hei 1, 390]. Here

$$p_1(a_v) = p(0°, a_v) \tag{6.53}$$

is a function of the true argument a_v only, taken at the point where $c = 0°$, that is where the epicycle centre is at the apogee of the deferent where it has a maximum distance from the Earth. Putting $c = 0°$ in (6.46) we get $\rho_{max} = R + e$, so that we have from (6.49)

$$\tan p_1(a_v) = -\frac{r \sin a_v}{R + e + r \cos a_v} \tag{6.54}$$

Since $R + e = 60^p$ it follows that (6.54) is the same function as (6.29) if we here put $a = a_v$. In other words, $p_1(a_v)$ is nothing else than the prosthaphairesis function known from the First Model and already tabulated [IV, 10; Hei 1, 337].

The function $p_2(a_v)$ is defined by

$$p_2(a_v) = p(180°, a_v) \tag{6.55}$$

Thus it is the prostaphairesis angle as a function of a_v only, taken at a point where $c = 180°$, that is at the perigee of the deferent where we have $\rho_{min} = R - e$. Accordingly (6.55) is determined by

$$\tan p_2(a_v) = -\frac{r \sin a_v}{R - e + r \cos a_v} \tag{6.56}$$

It is clear that for any given value of a_v (i.e. for a fixed position of the Moon on the epicycle relative to the true apogee A_v of the latter) we have the inequality

$$p_2(a_v) > p_1(a_v) \tag{6.57}$$

Therefore the term

$$\{p_2(a_v) - p_1(a_v)\} > 0 \tag{6.58}$$

represents the increase in the equation of argument when the epicycle centre moves from apogee to perigee of the deferent, the Moon being in a fixed position on the epicycle.

Formula (6.52) asserts, therefore, that the equation of argument corresponding to a position of the epicycle centre between the apogee and the perigee of the deferent can be found as the equation at the apogee, corresponding to the true argument a_v, plus a certain fraction of the total increase of the equation of argument from apogee to perigee, corresponding to the same true argument. This fraction is determined by the interpolation function f(c) which we now have to determine.

Let us, for a given position of the epicycle defined by a particular value of c, determine the maximum value of $p(a_v, c)$, treating c as constant and a_v as variable. This value can be denoted max p(c) and represents the greatest angle under which the radius of the epicycle is seen from the centre of the Earth at this particular position of the epicycle. Ptolemy then defines (cf. 3.58)

$$f(c) = 60 \cdot \frac{\max p(c) - \max p(0°)}{\max p(180°) - \max p(0°)} \tag{6.59}$$

This method of expressing $p(a_v, c)$ as the sum (6.52) is stated by Ptolemy without any proof and without any examination of how good the approximation is. That the error is, in fact, very small has been proved by Petersen in the paper mentioned above (page 159).

The General Table of Equations

The result of the preceding considerations is a general table of equations [V, 8; Hei 1, 390] of which the beginning is reproduced here:

I	II	III	IV	V	VI	VII
6°	354°	0°;53	0°;29	0°;14	0°;0,12	4°;58
12°	348°	1°;46	0°;57	0°;28	0°;0,24	4°;54

Columns I and II are arguments of the functions in the rest of the table; what they represent depends on which function they belong to. As usual they are given with intervals of 6° from 0° to 90°, and with intervals of 3° from 90° to 180°.

Column III contains the tabulated values of the function q(c) given by (6.47), that is the equation of centre. Here the arguments in Column I–II must be understood as values of the centrum (or double elongation) of the Moon.

Column IV is the function $p_1(a_v)$ given by (6.54). Accordingly the tabulated values are identical with those of the prosthaphairesis table [IV, 10; Hei 1, 337] of the First Model. But now the arguments in Columns I–II must be interpreted as values of the true argument a_v.

Column V contains the function (6.58), so that here too the arguments are values of the true argument a_v.

Finally Column VI tabulates the interpolation function $f(c)$ given by (6.59). In this case the arguments of Columns I–II are again values of the centrum c of the Moon.

The Simplified Calculation of Longitudes

By means of the general table of equations the programme for determining $\lambda(t)$ can now be described as follows [cf. V, 9; Hei 1, 392].

1°–5° are the same operations as in the programme described page 195.
6° Find $q(c)$ from Column III of the tables.
7° Find the true argument a_v by (6.45).
8° Find $p_1(a_v)$ from Column IV.
9° Find $\{p_2(a_v) - p_1(a_v)\}$ from Column V.
10° Find $\frac{f(c)}{60}$ from Column VI.
11° Multiply the two latter numbers, and
12° Add the product to the value $p_1(a_v)$ found in 8° to obtain $p(c, a_v)$.
13° Finally, add $p(c, a_v)$ to $\lambda_m(t)$ in order to find $\lambda(t)$, cf. (6.8).

Although there are more steps than before, there are no trigonometrical computations left, and the problem of finding the true longitude of the Moon can be solved exclusively by means of tables and simple arithmetical operations with tabulated values.

The question of how accurate this theory is has been discussed by several authors. Here we must refer to the paper by Viggo M. Petersen, showing by numerical analysis that the mean error in longitude is smaller than 1°;24 in Model III (which is much better than Model II or Model I). This is still more than two lunar diameters, one of the main causes of this discrepancy being the third anomaly of the Moon, i.e. the variation, which was first discovered by Tycho Brahe[9]). It has a maximum value of 0°;40. But on the whole Ptolemy and most of his Mediaeval followers found the theory to be sufficiently exact, and considering the inaccuracy of observation it must be acknowledged that the theory did fairly well. The only point where it fails completely is in giving the apparent size of the Moon. According to Model III the maximum and minimum distances of the Moon from the Earth are, respectively

$$\Delta_1 = R + e + r = 65^p;15 \tag{6.60}$$

$$\Delta_4 = R - e - r = 34^p;7 \tag{6.61}$$

(cf. (6.51) and (6.46)). This implies that the visible size of the Moon would double from syzygy to quadrature since the apparent diameter should vary roughly with a factor

9) On the discovery of the variation, see Thoren (1967). In 1836 L. Sédillot stated that the variation had been found already in the 10th century by the Baghdad astronomer Abu'l Wafa. This gave rise to much controversy between on the one hand Sédillot and Chasles, supported by Kempf (1878 p. 38), and, on the other, Munk, Biot, and Bertrand; see Dreyer 1906, p. 252 ff. Tycho's priority has now been established beyond any doubt.

2. Actually, it lies between 0°;29,22 and 0°;33,31. Curiously enough, Ptolemy nowhere comments upon this enormous discrepancy between theory and observation, although the corresponding values of Δ_1 and Δ_4 are clearly stated in the Almagest [V, 17; Hei 1, 429].

The Inclination of the Lunar Orbit

Until now we have considered the various models developed in this chapter as plane mechanisms located in the plane of the ecliptic. We must now remember that this was only a convenient approximation (see page 168). Actually all the circles of the models are placed in a plane having an inclination $i = 5°$ (the maximum latitude of the Moon) relative to the plane of the ecliptic. The two planes intersect each other along the nodal line from ☊ (the ascending node) through the centre of the Earth to ☋ (the descending node). The real situation is shown in Figure 6.15 where we see the ecliptic and the great circle defined on the heavenly sphere by the inclined plane of the lunar orbit. Let ☊ have the ecliptic longitude λ_n, and let us further consider a point of time t when the Moon has the true ecliptic longitude $\lambda = \lambda(t)$ as determined by the theory of longitudes explained above.

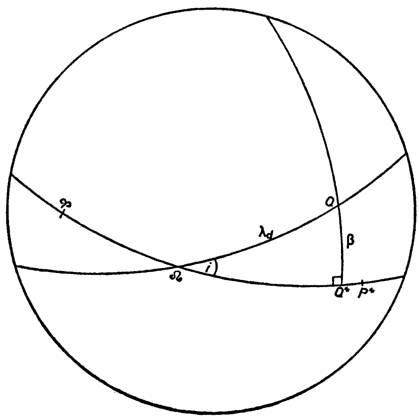


Fig. 6.15

This longitude is equal to the arc ♈P* from the vernal equinox to a certain point P* which we have until now taken as the projection on the ecliptic of the true centre of the Moon. It has the distance

$$\lambda_d = \lambda - \lambda_n \tag{6.62}$$

from the ascending node ☊. However, since the lunar orbit lies in the inclined plane the true position of the Moon must be represented by a point Q on the inclined circle

with the same distance λ_d from ☊. This point Q is projected upon the ecliptic in Q*. It follows that the approximation used in the theory of longitudes consists in identifying Q* and P*. This introduces an error $\delta\lambda$ in the computed longitude equal to the arc Q*P*, or

$$\delta\lambda = \lambda_d - ☊Q^*$$

where ☊Q* can be found from the right-angled triangle ☊Q*Q by the relation

$$\tan ☊Q^* = \tan ☊Q \cdot \cos i = \tan \lambda_d \cdot \cos i$$

This gives for the error

$$\tan \delta\lambda = \frac{\tan \lambda_d (1 - \cos i)}{1 + \tan^2 \lambda_d \cdot \cos i} \tag{6.63}$$

At the nodes we have $\lambda = \lambda_n$ and $\delta\lambda = 0$. To determine the maximum value of $\delta\lambda$ we find

$$\frac{d(\tan \delta\lambda)}{d\lambda_d} = \frac{(1 - \cos i)(1 - \tan^2 \lambda_d \cdot \cos i)}{\cos^2 \lambda_d (1 + \tan^2 \lambda_d \cos i)^2}$$

which shows that $\delta\lambda$ has an extremal value for

$$\tan^2 \lambda_d = \frac{1}{\cos i}$$

With $i = 5°$ this gives

$$\max \delta\lambda \approx 0°;7 \quad \text{for} \quad \lambda_d = \pm 45°;3 + p \cdot 180°$$

Thus the greatest error introduced by the 'reduction to the ecliptic' is smaller than one fourth of the apparent diameter of the Moon. This means that it is insignificant compared to the much greater mean error of the theory of longitudes mentioned above (page 198).

The Latitude of the Moon

To complete the theory of the Moon's motion we must now examine how Ptolemy calculates its latitude $\beta(t)$ at a given time t. The procedure is ascribed to Hipparchus [V, 7; Hei 1, 388] and there is no reason to doubt this since the latter must have had a theory of latitudes in order to be able to undertake his work on eclipses. The method is of great simplicity compared with the intricate theories of latitude of the other planets, and is very briefly described in a short section of the chapter in which Ptolemy explains the construction of the general table of equations [V, 7; Hei 1, 388]. Here Ptolemy says only that in order to find $\beta(t)$ one has to proceed in the same way as

when determining the declination of the Sun (see page 95). This means that $\beta = Q^*Q$ is found from the triangle $☊ Q^*Q$ in Figure 6.15 by the relation

$$\sin \beta = \sin i \cdot \sin \lambda_d \qquad (6.64)$$

or, since both β and its maximum value i are small angles

$$\beta = i \cdot \sin \lambda_d \qquad (6.65)$$

Here (6.64) is analogous to the formula (4.2) for the declination of a point on the ecliptic. This explains the reference to the declination of the Sun.

In the general table of equations [V, 8; Hei 1, 390] the final Column VII contains the tabulated values of a function

$$\beta = 5° \cdot \sin (\lambda_\beta + 90°) \qquad (6.66)$$

corresponding to the argument λ_β in Columns I and II. This function is identified with the latitude of the Moon, which implies that

$$\lambda_\beta + 90° = \lambda_d$$

or

$$\lambda_\beta = \lambda_d - 90° \qquad (6.67)$$

Thus the argument of the tabulated latitude function is reckoned, not from the ascending node, but from a point 90° further to the East, that is from the top, or highest northern point, or the northern limit, of the inclined circle.

The procedure of finding the latitude at a given time appears clearly from (6.62), (6.66), and (6.67). We must only remember that the nodes are continually moving in the retrograde direction with a daily motion of

$$\omega_d - \omega_t \approx 0°;3 \text{ per day}$$

relative to the vernal equinox. This means that the longitude of the ascending node is found as the difference

$$\lambda_n(t) = \lambda_n(t_0) - (\omega_d - \omega_t). (t - t_0) \qquad (6.68)$$

It is, perhaps, not beside the point to call attention to the fact that the retrograde motion of the nodes has nothing to do with the retrograde movement of the deferent in the theory of longitudes[10]).

Conclusion

Here ends Ptolemy's lunar theory of the longitude and latitude of the Moon. There follows a chapter [V, 10; Hei 1, 394] in which he refutes any suspicion that the transition from Model I to Model II and further to Model III would spoil the agree-

10) Cf. the erroneous statement by Duhem, *Système du Monde* (I, p. 495).

ment between theory and observation at the syzygies upon which Model I was founded. We shall not summarize this chapter here, but only notice as a final remark that the preceding theories describe everything which Ptolemy knows of the motion of the Moon – apart from phenomena depending upon the knowledge of the absolute distance of the Moon (e.g. its parallax), or of the Sun (eclipses). These questions are dealt with in the remaining part of Book V and in Book VI.

CHAPTER 7

Parallaxes and Eclipses

Introduction

The theories of the Sun and Moon developed in the preceding chapters have the feature in common that they provide exactly geocentric co-ordinates. In other words they presuppose an ideal observer situated at the centre T of the Earth and observing the centre of the planet in the direction TP (see Figure 7.1). A real observer

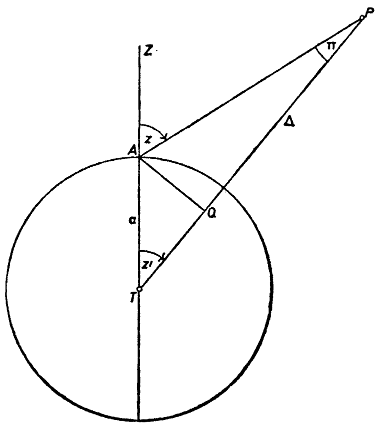


Fig. 7.1

placed at the point A on the surface of the Earth will see P in the direction AP. Unless P is at the zenith of A, the lines of vision TP and AP will include an angle called the parallax of P (in modern terms the daily parallax). It is the difference

$$\pi = z - z' \tag{7.1}$$

between the observed and the geocentric zenith distances of P. If we want to know the actual position of P (the direction of AP) then it is necessary to adjust the geocentric position derived from theory with a certain correction for parallax.

In general such corrections will be small, and for most of the heavenly bodies unobservable with the instruments at Ptolemy's disposal. According to the Almagest neither the fixed stars nor the five ordinary planets show any appreciable parallax. On the other hand the Moon is near enough to the Earth to make its observed position differ from the theoretical place to a very considerable extent. Whether this be the case for the Sun too remains to be seen.

The existence of a lunar parallax means, first, that we have to correct some of the

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_7,

observations of the Moon upon which the lunar theory was founded. This was actually done by Ptolemy, although in the preceding chapter we did not explain his method in detail.

Next, a quantitative theory of the parallax of the Moon is essential for the theory of eclipses, and in particular for the determination of the geographical areas from which eclipses of the Sun are visible[1]). Now the determination of eclipses was one of the main purposes of ancient astronomy. This was so mainly for astrological reasons; but it is worth remembering that Hipparchus underlined the importance of lunar eclipses also for the determination of geographical longitudes.

Thus the Almagest would be incomplete without the theory of eclipses which is found in Book VI, with the parallax theory in Book V, Chapters 11–19 as a necessary prerequisite. It appears that Ptolemy took over most of the latter theory from Hipparchus, to whose (lost) book *On the Computation of Parallaxes* he refers [V, 19; Hei 1, 450]. In the present chapter we shall consider the essential features of both these theories, without going into such detail as in the exposition of the theory of the Moon.

The main lines of the parallax theory are the following: First the parallax of the Moon is found as the difference between a measured and a theoretical position. From this the distance of the Moon from the Earth is derived. A similar method is impossible in the case of the Sun since this body shows no measurable parallax. It appears from what Ptolemy says [V, 11; Hei 1, 402] that Hipparchus had tried to find the distance of the Sun from the rather vague assumption that its parallax is *just great enough to be perceptible*, whatever that may mean. Ptolemy succeeds in avoiding such imprecise assumptions, showing that it is possible to find the value of at least one linear distance connected with the distance of the Sun, viz. the width of the Earth's shadow, which can be derived from the magnitude of a lunar eclipse. When that is found, the distance of the Sun follows from simple geometrical considerations, and the solar parallax is easily computed, although with a result which is far from being true, just as the method itself gives rise to serious misgivings. We shall now follow these various steps in more detail.

The Parallax of the Moon

The first step consists in the construction of a special instrument of observation, called the *parallactic ruler*[2]), and designed for the purpose of measuring zenith

1) It is remarkable fact that no actual eclipse of the Sun is recorded in the *Almagest* which contain detailed records of at least 21 lunar eclipses (see Appendix A). According to Neugebauer (1951, p. 89) there are no records at all of solar eclipses observed in Egypt before A.D. 601 (cf. Allen 1947). That the phenomenon itself was known goes without saying (cf. below note 4).

2) Contrary to what he used to do Proclus (*Hyp.* IV, 2, 49) did not describe the details of the parallactic instrument, finding it impossible to improve upon the description in the *Almagest*. The instrument was the simplest astronomical precision instrument in Antiquity, consisting of only a vertical pole with a plumb line and two straight rulers, cf. Dicks 1954, p. 80 f. It conserved its popularity during the Middle Ages, and was used also by Copernicus, whereas Tycho Brahe had a poor opinion even of his own much more sophisticated version of the instrument (see Thoren 1973).

distances with great precision [V, 12; Hei, 403]. Only one observation with this instrument is quoted here. It was performed by Ptolemy in Alexandria in the year Hadrian 20 on Athyr 13 at sunset (about 5^h50^m) when the Moon was very near the meridian. Its apparent (visible) zenith distance was $z = 50°;55$. This empirical value must now be compared with the true, geocentric zenith distance z', derived from the Lunar theory in the following way:

From the standard epoch to the time of the observation there is an interval of time equal to

$$t - t_0 = 882^a 72^d 5^h 20^m$$

From the theory of the Sun we can then find the

mean longitude of the Sun $\lambda_{m_\odot} = 187°;31$

true longitude of the Sun $\lambda_\odot = 185°;28$

while from the lunar theory we can calculate the

mean longitude of the Moon $\lambda_{m_☽} = 265°;44$

mean argument of the Moon $a_m = 262°;20$

equation of the Moon $p = 7°;26$

true longitude of the Moon $\lambda_☽ = 273°;10$

The true longitude of the Moon is determined merely in order to enable us to find the Moon's latitude. Here we have to find the position of the ascending node ☊ at the time of the observation from (6.68). Adding 90° we have the longitude of the highest northern point of the deferent, which subtracted from the true longitude of the Moon gives the true argument of latitude as 2°;6. From the latitude table [V, 8; Hei 1, 390] we find that at the time of observation the Moon had a northern latitude $\beta = 4°;59$. At the position here considered this is very nearly equal to the distance of the Moon from the ecliptic measured along the meridian.

We now use the table of the obliquity of the ecliptic [I, 15; Hei 1, 80] with the argument $\lambda = 273°;10$ and find the corresponding southern declination of the Moon $\delta = 23°;49$. Furthermore, at Alexandria the meridian point of the equator is (er-

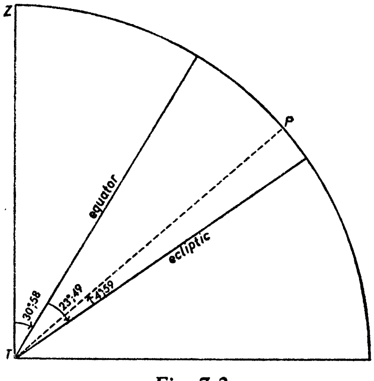


Fig. 7.2

roneously) supposed to have a zenith distance of $\varphi = 30°;58$ (see page 109). By means of Figure 7.2 we find the zenith distance of the Moon as

$$z' = 30°;58 + 23°;49 - 4°;59 = 49°;48$$

This gives the lunar parallax in altitude

$$\pi = z - z' = 1°;7$$

From this follows a horizontal parallax of 1°;26 (see Delambre II, p. 210). This is a very erroneous result. The modern value of the mean horizontal parallax is 0°;57, and the correct result of Ptolemy's observation would have been about 0°;44. The error in φ is partly responsible for this; the more correct value $\varphi = 31°;13$ would give $\pi = 0°;52$ which is still too much; the remaining error is presumably due to faults of the instrument, or of the calculated latitude, or both. But since Ptolemy does not question the accuracy of his result the rest of the theory of parallaxes and eclipses is founded upon the above value $\pi = 1°;7$.

The Distance of the Moon

Let us again consider Figure 7.1. Here the circle represents a cross section of the Earth through the meridian at the point A where the observation was made. The observed zenith distance z is the angle ZAP, and the theoretical, geocentric zenith distance z' is the angle ZTP. A line through A meets TP perpendicularly in Q. Simple trigonometric considerations show that

$$\begin{aligned} TQ &= a \cdot \cos z' \\ QA &= a \cdot \sin z' = AP \cdot \sin \pi \\ QP &= AP \cdot \cos \pi = a \cdot \sin z' \cdot \cot \pi \end{aligned}$$

The distance between the centres of the Earth and the Moon is

$$\begin{aligned} \Delta &= TP = TQ + QP \\ &= a\{\cos z' + \sin z' \cdot \cot \pi\} \end{aligned} \tag{7.2}$$

or, expressed in terms of the observed zenith distance z

$$\Delta = \frac{a \sin z}{\sin \pi} \tag{7.3}$$

With $a = 1^r$ = the radius of the Earth, $z' = 49°;48$, and $\pi = 1°;7$ Ptolemy finds

$$\Delta = 39^r;45$$

This result makes it possible to find absolute values of the linear parameters of the lunar theory, for, applying the observational data (6.51) we find

$$\Delta = 40^p;25$$

expressed in the arbitrary unit 1^p. Consequently we have

$$40^p;25 = 39^r;45$$

from which follows

$$1^p : 1^r = 39;45 : 40;25 = 477 : 485 \approx 59 : 60 \tag{7.4}$$

In order to express the parameters in units of 1^r we must accordingly reduce their previously found values (in units of 1^p) in the ratio 59/60. This gives

	Unit 1^p	Unit 1^r
r = radius of epicycle	$5^p;15$	$5^r;10$
R = radius of deferent	$49^p;41$	$48^r;52$
e = eccentricity	$10^p;19$	$10^r;8$
R + e = mean distance	$60^p;0$	$59^r;0$

As already mentioned (page 198) the lunar theory leads to the erroneous consequence that the apparent diameter of the Moon should vary by a factor 2 during a complete lunation. It is thus a matter of opinion whether one should say that Ptolemy found a satisfactory value of the distance of the Moon. On the other hand his value of the mean distance of the epicycle centre $R + e = 59^r$ is near to the mean distance of the Moon $60^r.27$ accepted to-day. It should be noticed that Ptolemaic value agrees fairly well with what earlier astronomers had found. Thus Hipparchus is credited with the value of $67^r;20$ and Posidonius with $52^r;24$. This reveals the progress of Greek astronomy since the time of Aristarchus, who found a distance of $19^r;0$ only (Heath 1913, p. 350).

On the other hand the small denominator of (7.3) means that Δ is very susceptible to small changes of π. If the zenith distance is not too small we can consider sin z as a constant when differentiating (7.3), which gives

$$\delta\Delta = -\Delta \cdot \cot \pi \cdot \delta\pi \tag{7.5}$$

A rough calculation leads to the following values

$\delta\pi$	π	$\delta\Delta$
$-0°;1$	$1°;6$	$+0^r;36$
$-0°;10$	$0°;57$	$+7^r;0$

which illustrate the effect on Δ of plausible observational errors in Ptolemy's value of π.

The Apparent Size of the Moon

In order to find the size of the Moon (regarded as a spherical body) relative to that of the Earth we must know its apparent angular diameter, which is roughly $\frac{1}{2}°$.

To obtain a more precise value Ptolemy constructed another special instrument, originally invented by Hipparchus and called the *dioptra*[3]) [V, 14; Hei 1, 417]. Observing with this instrument he found that

1) The apparent diameter of the Sun is constant within the limits of observation. This tells us something about the quality of the instrument, since we know that the solar diameter varies between 0°;32,36 and 0°;31,31.

2) That the apparent diameter of the Moon is variable and equal to that of the Sun when the Full Moon is at the apogee of the epicycle, i.e. at the greatest distance from the Earth[4]). Actually the lunar diameter varies between 0°;33,31 and 0°;29,22.

Nothing more could be established by means of the dioptra, and in order to find the exact size of the diameter of the Moon Ptolemy had to devise another and more indirect method based on two old lunar eclipses observed in Babylon [V, 14; Hei 1, 418].

1) The first eclipse (Appendix A, N° 4) began about 16^h45^m on the 27/28 Athyr in the 5th year of Nabopalassar (621 B.C. April 22). It reached its maximum about one hour later when one quarter of the lunar diameter was eclipsed.

2) The second eclipse (Appendix A, N° 5) took place at 11^h on Phamenoth 17/18 in the 7th year of Kambyses (523 B.C. July 16) when one half of the Moon was eclipsed.

These eclipses are now reduced to the meridian of Alexandria and the exact positions of the Moon found by means of the lunar theory developed in Chapter 6. We must here omit the actual calculations and only mention the results, viz. that at the maximum of the first eclipse the centre of the Moon had an angular distance from the descending node ☋ of 9°;20 whereas it at the maximum of the second eclipse had a distance of 7°;48 from the ascending node ☊. The two situations are reproduced in Figures 7.3 a–b where the shaded area represents a cross section through the shadow of the Earth at the time of the maximum eclipse.

3) Ptolemy discards older methods as unreliable, for instance that of measuring the rising time of the Sun at the equinoxes (when the Sun is upon the equator) by means of a water clock. This method is described in great detail by Proclus (*Hyp.* IV, 3, 73 f.) who also gave an account of the construction and use of the dioptra (ibid., 87 f.). – Cf. Dicks 1954.

4) The assertion that the apparent diameter of the Moon is greater than, or equal to that of the Sun would – if it were correct – preclude the possibility of annular eclipses of the Sun. The implication is that Ptolemy was unaware of the existence of such eclipses of which no records seem to have been made in Antiquity. This is a curous fact, considering that the phenomenon could have been observed, and that it was fully understood and described in general terms. First, an eclipse dated to – 477 Feb 16 was annular at Sardes in Asia Minor; this was calculated by Zech (1853, p. 30) who tentatively identified it with the famous eclipse preceding Xerxes' crossing of the Hellespont and recorded by Herodotus (VII, 37) as total. Second, 200 years or more before Ptolemy the Alexandrian astronomer Sosigenes explained the mechanism of annular eclipses in a fragment, preserved by Simplicius (*De caelo*, ed. Heiberg, p. 504 f., cf. the translation by Heath, 1913, p. 221 f.). Before Simplicius, Proclus (*Hyp.* I, 19–20) described the phenomenon which he regarded as evidence of the varying distance of the Moon, but not of the Sun.

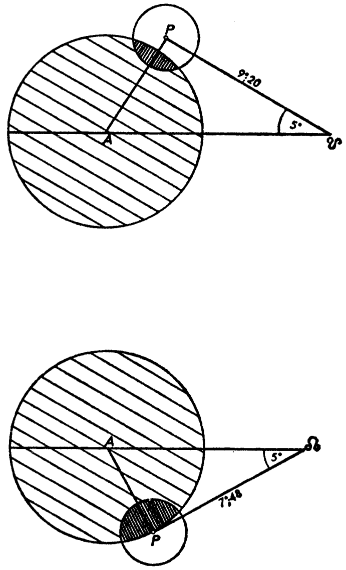


Fig. 7.3 a–b

From the two spherical triangles defined by the centre A of the shadow, the centre P of the Moon, and the node we find

for the first eclipse AP = 0°;48,30
for the second eclipse AP = 0°;40,40

The difference 0°;7,50 is one quarter of the diameter of the Full Moon which accordingly is 0°;31,20, corresponding to a radius of 0°;15,40.

For the size of the shadow at the distance of the Moon we have therefore

$$\frac{\text{radius of shadow}}{\text{radius of Moon}} = \frac{0°;40,40}{0°;15,40} = 2\tfrac{3}{5} \tag{7.6}$$

Without being more specific Ptolemy maintains that *a great number of similar observations gave very much the same result*[5]) [V, 14; Hei 1, 421].

The Distance of the Sun

We have mentioned already (page 204) that Hipparchus at first nourished the idea that the Sun had a 'barely perceptible' parallax from which its distance could be deduced, after which the distance of the Moon would follow by means of observations of eclipses [V, 11; Hei 1, 402]. Later he discarded this hypothesis and attacked

5) This is one of the points where Ptolemy tried to improve on Hipparchus who, according to Pappus (*Math. Coll.*, ed. Hultsch p. 557) found a ratio of $2\frac{1}{2}$.

the problem from the other end, starting with the distance of the Moon and inferring that of the Sun [V, 14; Hei 1, 421]. This latter method (cf. Swerdlow, 1969) was adopted by Ptolemy, who never tried to measure the solar parallax directly, as far as we can see from the Almagest. Instead he showed how a theoretical value of the distance $\Delta_{\odot}$ of the Sun can be determined from the more or less empirical data quoted above, viz. the distance of the Moon, its apparent diameter, and the ratio of the latter to the diameter of the shadow of the Earth given by (7.6) [V, 15; Hei 1, 422].

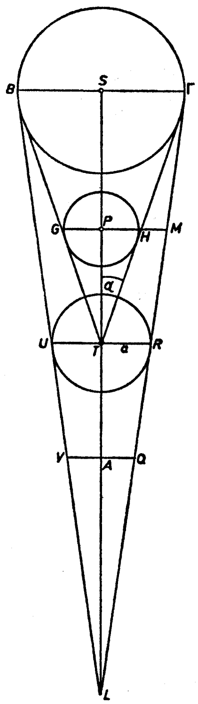


Fig. 7.4

The basic principle can be understood from Figure 7.4, where the centre of the Earth, Moon, and Sun are denoted by T, P, and S respectively. A and P are equally distant from T so that TA = TP = Δ is the distance of the Moon from the Earth. The apparent radius of the Moon is α. By means of the values of Δ and α we can construct the lines TH and TG which are tangents to the circle representing the Moon. They must also be tangents to the Sun, if the figure corresponds to a situation in which the Full Moon is at the apogee of the epicycle, according to the alleged observational fact mentioned above (page 208). Since we know the diameter VQ of the Earth's shadow from (7.6) we can draw the tangents QR and VU to the Earth. These lines are also tangents to the Sun which casts the shadow. Therefore, the points B and Γ of the two sets of tangents must define a chord of the Sun and hence deter-

mine the position of S and its distance $TS = \Delta_{\odot}$ from the Earth. Thus the problem is solved geometrically. We notice that since the angles are small we commit no great error in supposing the lines UR, GH and BΓ to be diameters of their respective circles.

In the Almagest, $\Delta_{\odot}$ is found by a numerical calculation which can be generalized in the following way. In the triangle TPH we have approximately $TH \approx TP = \Delta$, and consequently

$$PH = \Delta \cdot \sin \alpha$$

From (7.6) we can then find

$$AQ = \tfrac{13}{5} \cdot \Delta \cdot \sin \alpha \tag{7.7}$$

Since the radius of the Earth TR is a parallel transversal in the trapezium AQMP we have

$$TR = 1^r = \tfrac{1}{2}(PM + AQ)$$

whence

$$\begin{aligned} PM &= 2^r - AQ \\ &= 2^r - \tfrac{13}{5} \cdot \Delta \cdot \sin \alpha \end{aligned}$$

We have further

$$\begin{aligned} HM &= PM - PH \\ &= 2^r - \tfrac{18}{5} \cdot \Delta \cdot \sin \alpha \end{aligned}$$

From the similar triangles TRΓ and HMΓ it follows that

$$\frac{T\Gamma}{H\Gamma} = \frac{TR}{HM}$$

Since $TR = 1^r$, $T\Gamma \approx TS$ and $H\Gamma \approx PS = TS - TP = \Delta_{\odot} - \Delta$ this proportion can be approximated by

$$\frac{\Delta_{\odot}}{\Delta_{\odot} - \Delta} = \frac{1^r}{2^r - \tfrac{18}{5} \cdot \Delta \cdot \sin \alpha}$$

which is solved by

$$\Delta_{\odot} = \frac{\Delta}{\tfrac{18}{5} \cdot \Delta \cdot \sin \alpha - 1^r} \tag{7.8}$$

(cf. the more general formula in Swerdlow 1972, p. 97). Using the values $\Delta = 64^r;10$ and $\alpha = 0°; 15, 40$ deduced above, Ptolemy finds the distance of the Sun

$$\Delta_{\odot} = 1210^r$$

Since we know that the actual value is about 23400^r it follows that the Ptolemaic value is about 19 times too small. Compared with his fairly correct distance of the

Moon this is a rather poor result. That it is also a somewhat fortuitous result appears from an estimate of the uncertainty of the calculation.

We saw from (7.5) that an error of a few minutes of arc in the lunar parallax will alter the distance Δ of the Moon by several units (page 207). If we keep the apparent diameter of the Moon constant but alter its distance to $\Delta = 63^r$ we find from (7.8) a solar distance of $\Delta_\odot = 1903^r$. Furthermore, the distance of the Sun would be infinite for a value of Δ determined by putting the denominator in (7.8) equal to zero:

$$\tfrac{18}{5} \cdot \Delta \cdot \sin 0°;\ 15{,}40 - 1 = 0$$

which gives $\Delta = 60^r; 59$.

The latter 'infinity' value of the lunar distance Δ is only $3^r;11$ smaller than that used by Ptolemy, which corresponds to a diminution of the lunar parallax by only about $\delta\pi = -0°;3$; this is presumably within the expected error of the parallax. Accordingly we must conclude that Ptolemy could easily have found a widely different value of the distance of the Sun.

This is not stated in the Almagest; but in the newly discovered part of the *Hypotheses* it is clearly said that if we increase the distance to the Moon slightly, the distance to the Sun will be somewhat diminished (see Goldstein 1967, p. 7).

The question why Ptolemy adhered to the value $\Delta_\odot = 1210^r$ without questioning its credibility is not easily answered. That he should have used the parallaxes determined by his predecessors to calculate faked observations upon which he could then construct his own theory seems a very far fetched idea, which even Delambre does no more than hint at. A more likely explanation is that he was well aware of the difficulty of the problem and the uncertainty of the result, but for that very reason wished to keep as near as possible to the value of $\Delta_\odot$ found by Aristarchus, who proved that

$$18\,\Delta < \Delta_\odot < 20\Delta$$

where Δ is the distance of the Moon (Heath, 1913, p. 377). Using the value $\Delta = 64^r;10$ determined above, this gives

$$1155^r < \Delta_\odot < 1283^r;20$$

with a mean value of $1219^r;10$ which is only slightly different from Ptolemy's.

Finally, it seems probable that Ptolemy did not attach much importance to the problem of the distance of the Sun. Thus with $\Delta_\odot = 1210^r$ we find from (7.3) that the corresponding horizontal parallax is too small to be of much practical interest (see page 214). Another fact pointing to the same conclusion is that Ptolemy does not trouble to tell us whether $\Delta_\odot = 1210^r$ is the maximum, minimum, or mean distance.[6]). In fact, later he treats the distance of the Sun as a constant regardless of the eccentricity of its orbit (see page 214). Actually he considered 1210^r as the mean distance, at any

6) Already in Antiquity this lack of precision gave rise to misunderstandings. Thus Proclos stated (*Hyp*. VII, 20) that Ptolemy found 1210^r as the greatest distance of the Sun. In his commentary on the Almagest (V, 15; ed. Rome, p. 107–108) Pappus gave the minimal, mean, and maximal distances as $1159^r;35$, 1210^r, and $1260^r;25$ respectively.

rate in the *Hypotheses*, where he states that the maximum, mean, and minimum distances are 1260^r, 1210^r and 1160^r respectively, in agreement with his usual value 1/24 of the eccentricity (page 147).

Having found the distances of the Moon and Sun and their apparent (angular) diameters, Ptolemy devotes a particular chapter to a calculation of their actual sizes [V,16; Hei 1, 426]. From Figure 7.4 we find the radius of the Sun from the proportion

$$\frac{S\Gamma}{PH} = \frac{TS}{TP}$$

to be

$$S\Gamma = 5^r;30$$

where the radius of the Moon was computed above as

$$PH = 0^r;17,33$$

Ptolemy also interprets these values in the manner that the diameters of the Earth and the Sun are $3\frac{2}{5}$ and $18\frac{4}{5}$ times as long as that of the Moon. Also their relative volumes are calculated. Of interest to the theory of eclipses is the computation of the length TL of the shadow of the Earth. We have

$$\frac{TR}{AQ} = \frac{TL}{AL} = \frac{TL}{TL - TA}$$

where $TR = 1^r$, $AQ = 0^r;45,38$ and $TA = TP = 64^r;10$. The calculation [V, 15; Hei 1, 425] gives

$$TL = 268^r$$

The General Formula of Parallax in Altitude

Having found the geometric parameters of both the solar and lunar theories in units of 1^r, Ptolemy is now ready to tackle the computation of parallaxes in general. He begins [V, 17; Hei 1, 428] with calculating the parallactic displacement π (z′) along a vertical circle through the star and the zenith of the observer (the parallax in altitude). The method is, as usual, given in the form of a numerical example; but the result corresponds to a formula which can be derived by solving (7.2) with respect to π

$$\tan \pi(z', \Delta) = \frac{a \sin z'}{\Delta - a \cos z'} \tag{7.9}$$

and substituting $\sin \pi$ for $\tan \pi$, giving

$$\sin \pi(z', \Delta) = \frac{a \sin z'}{\Delta - a \cos z'} \tag{7.10}$$

Here a is the radius of the Earth, Δ the distance of the star, and z' the geocentric zenith distance. We shall call (7.10) the fundamental formula of parallax, since all the following calculations are founded upon it. It would have been simpler to use the observed zenith distance z and derive the parallax from (7.3) as

$$\sin \pi(z, \Delta) = \frac{a \sin z}{\Delta} \tag{7.11}$$

but Ptolemy's procedure shows that he has the geocentric zenith distance in mind; this is understandable if we remember that in most of the following cases the zenith distance will be determined not from observations, but from theoretical computations of geocentric coordinates.

In the case of the Sun the distance Δ is supposed to be constant $\Delta = 1210^{r}$. This gives the formula of the solar parallax:

$$\sin \pi(z') = \frac{\sin z'}{1210 - \cos z'} \tag{7.12}$$

from which Ptolemy computes $\pi(z')$ for intervals of 2° of z' from 0° (zenith) to 90° (horizon). These values are tabulated in column 2 of a general table of parallax [V, 18; Hei 1, 442].

Ptolemy deals with the parallax of the Sun as if the eccentricity of the solar orbit were zero. This is, of course, a crude approximation which seems strange in view of the actual value of $e = \frac{1}{24}$. That he omits to develop a more refined theory may very well be for the practical reason that the maximum value $\pi(90°) = 0°;2,52$ is so small that observations with the instruments at Ptolemy's disposal were too inaccurate to give any importance to a possible correction of less than 3 minutes of arc. It should be noticed that the value of the horizontal solar parallax accepted in modern astronomy is 0°;0,8,48.

In the case of the Moon the situation is very different. First, the maximum lunar parallax is much greater than the usual errors of observation. Second, the distance Δ of the Moon from the Earth is not approximately constant, but a strongly varying function of the centrum c of the Moon and of its true argument a_v on the epicycle. This function is given by (6.51). In other words we have to deal with a parallax function (7.10) of three variables, expressed by

$$\sin \pi(z', c, a_v) = \frac{a \cdot \sin z'}{\Delta(c, a_v) - a \cos z'} \tag{7.13}$$

Accordingly the theory of the lunar parallax is mainly a problem of how to handle this particular function of 3 variables (cf. Chapter 3, page 89).

The Lunar Parallax as a Function of 3 Variables

Formula (7.13) shows that the zenith distance z' is the strongest of the 3 variables. According to Ptolemy's usual procedure with functions of more than one variable this

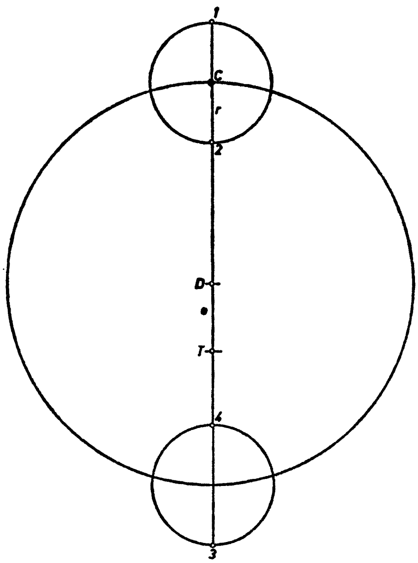


Fig. 7.5

means that we have to tabulate the parallax $\pi(z', c, a_v)$ as a function of z' only, giving suitable constant values to the centrum c and the true argument a_v. Ptolemy does this [V, 17; Hei 1, 429] for four standard positions of the Moon (cf. Figure 7.5) defined according to the following scheme in which the distances Δ_i are found by means of (6.51) and (6.46):

Position of the epicycle centre on the deferent	Position of the Moon on the epicycle	
	True apogee $a_v = 0°$	True perigee $a_v = 180°$
apogee $c = 0°$	$\Delta_1 = R + e + r = 64^r;10$	$\Delta_2 = R + e - r = 53^r;50$
perigee $c = 180°$	$\Delta_3 = R - e + r = 43^r;53$	$\Delta_4 = R - e - r = 33^r;33$

By means of (7.13) we can now define four standard partial functions

$$\begin{aligned}\pi_1(z') &= \pi(z', \Delta_1)\\ \pi_2(z') &= \pi(z', \Delta_2)\\ \pi_3(z') &= \pi(z', \Delta_3)\\ \pi_4(z') &= \pi(z', \Delta_4)\end{aligned} \qquad (7.14)$$

giving π as function of the zenith distance only at the four standard distances Δ_1, Δ_2, Δ_3, Δ_4 of the Moon from the Earth. Of these partial functions $\pi_1(z')$ and $\pi_3(z')$ are tabulated in Columns III and V respectively of the general table of parallax.

We must now take other than the four standard positions of the Moon into account. Ptolemy begins by keeping the centrum $c = 0°$ constant, considering only how the motion of the Moon on the epicycle at the apogee of the deferent affects the parallax. We see from the scheme above that now, when the true argument a_v increases from $0°$ to $180°$, the distance Δ will decrease by the amount $2r$ from Δ_1 to Δ_2, and the parallax increase from $\pi_1(z')$ to $\pi_2(z')$. At an intermediate position, Ptolemy postulates (without proof) that the parallax as a function of the two variables z' and a_v is given by the expression

$$\pi'(z', a_v) = \pi_1(z') + [\pi_2(z') - \pi_1(z')] \cdot \frac{f'(a_v)}{60} \tag{7.15}$$

where $f'(a_v)$ is a function given by

$$f'(a_v) = \frac{\Delta_1 - \Delta'}{2r} \cdot 60 \tag{7.16}$$

and the distance

$$\Delta' = \sqrt{(r \sin a_v)^2 + (R + e + r \cos a_v)^2} \tag{7.17}$$

is a function of a_v only found by putting $c = 0°$ in (6.51). The total increase of parallax from $a_v = 0$ to $a_v = 180°$ is

$$\pi_2(z') - \pi_1(z') \tag{7.18}$$

which is tabulated in Column IV of the parallax table, while the interpolation function $f'(a_v)$ is given in Column VII. Since the argument in Column I runs from $0°$ to $90°$, we must interpret these as double degrees when entering into Column VII, i. e. we must enter with the argument $28°$ if we want to find the function $f'(56°)$.

Finally we have to account for the influence on π of the third variable, the centrum c. Here the procedure is repeated once more. Let us consider a situation in which the epicycle centre is at the perigee of the deferent. A similar line of reasoning as before leads to the expression

$$\pi''(z', a_v) = \pi_3(z') + [\pi_4(z') - \pi_3(z')] \cdot f''(a_v) \tag{7.19}$$

Here the total increase of parallax from $a_v = 0°$ to $a_v = 180°$ is

$$\pi_4(z') - \pi_3(z') \tag{7.20}$$

which is tabulated in Column VI of the parallax table. The interpolation function

$$f''(a_v) = \frac{\Delta_3 - \Delta''}{2r} 60 \tag{7.21}$$

is found in Column VIII. The distance function Δ'' is here

$$\Delta'' = \sqrt{(r \sin a_v)^2 + (R - e + r \cos a_v)^2} \tag{7.22}$$

We have now found two parallax functions, viz.

$\pi'(z', a_v)$

valid at the apogee of the eccentric where $c = 0°$ and the distance $\rho(c) = TC$ of the epicycle centre from the centre of the Earth is $\rho(0°) = R + e$ (see 6.46); and

$\pi''(z', a_v)$

valid at the perigee where $c = 180°$ and $\rho(180°) = R - e$.
Corresponding to other values of c Ptolemy constructs a general parallax function corresponding to the expression

$$\pi(z', c, a_v) = \pi'(z', a_v) + [\pi''(z', a_v) - \pi'(z', a_v)] \cdot \frac{g(c)}{60} \tag{7.23}$$

which is again assumed without proof. Here the interpolation factor g(c) is constructed as

$$g(c) = 60 \cdot \frac{\rho(0°) - \rho(c)}{\rho(0°) - \rho(180°)} \tag{7.24}$$

or

$$g(c) = 60 \cdot \frac{R + e - \rho(c)}{2e} \tag{7.24a}$$

This function is tabulated in Column IX, the arguments in Column I being understood as values of the centrum c expressed in double degrees. Since the centrum c is twice the mean elongation of the Moon from the Sun, it follows that the arguments in Column I can be understood also as mean elongations when used in connection with Column IX.

The general arrangement of the parallax table is then the following

Column.........	I	II	III	IV	V	VI	VII	VIII	IX
Function........	z'	$\pi_\odot$	π_1	$\pi_2 - \pi_1$	π_3	$\pi_4 - \pi_3$	$f'(a_v)$	$f''(a_v)$	$g(c)$

The method of handling the general parallax function formalized above may seem rather cumbersome. Yet it represents something of a major victory over a very untractable problem. Since the table of parallax has 45 rows, a table with 3 entries would demand $45^3 = 91\,125$ tabulated values. Instead we have a table of the lunar parallax in which only $45 \cdot 7 = 315$ values have to be computed.

In the final chapter [V, 19; Hei 1, 444] Ptolemy begins by explaining how a lunar parallax in altitude is computed by means of the tables. The procedure is in accordance with (7.23), (7.19), and (7.15), and we are reminded that the zenith distance z' can be found from the tables in II, 13 [Hei 1, 174] if we know the ecliptic longitude of the Moon, its hour angle, and the geographical latitude (cf. page 119). Here Ptolemy

tacitly assumes that the Moon is on the ecliptic, i. e. that its latitude is zero. This is, of course, true only at the nodes, and therefore approximately true at eclipses. This indicates that the parallax theory is devised as a means of reducing eclipse observation and predicting eclipses.

The Parallax in Longitude and Latitude

We must now investigate how the parallax in altitude $\pi(z', c, a_v)$ determined above affects the ecliptical coordinates λ and β of a star. In Figure 7.6 S' is the geocentric position of the star and S its actual position as seen by an observer with the zenith Z.

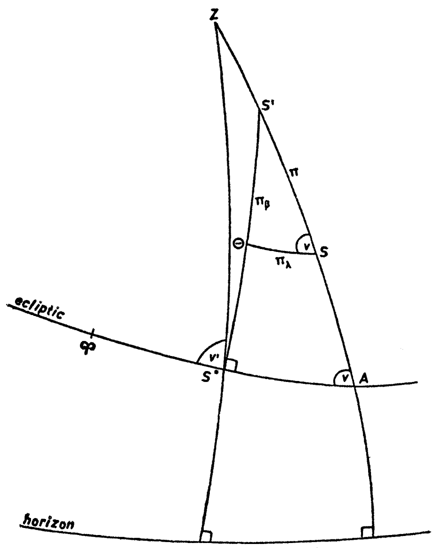


Fig. 7.6

Thus $ZS' = z'$ is the geocentric and $ZS = z$ the observed zenith distance, while $\pi = S'S$ is the parallax in altitude. The vertical through Z, S', and S intersects the ecliptic in A. A great circle through S' perpendicular to the ecliptic intersects the latter at S^*. Then $\Upsilon S^* = \lambda$ and $S^* S' = \beta$ are the geocentric longitude and latitude respectively. A small circle through S parallel to the ecliptic intersects $S'S^*$ in Θ. Then $S'S\Theta$ is the parallax triangle in which

$\pi \;= S'S$ is the parallax in altitude
$\pi_\lambda = S\Theta$ is the parallax in longitude
$\pi_\beta = S'\Theta$ is the parallax in latitude

In order to find the two latter components we must know one more angle in the parallax triangle.

Here Ptolemy begins with an approximative method, used already by Hipparchus in the (lost) book *On the Computation of Parallaxes* to which he refers [V, 19; Hei 1, 450]. The method rests on the assumption that the

$$\text{angle } S'S\Theta = \text{angle } ZAS^* = v$$

is equal to the

$$\text{angle } ZS^*\Upsilon = v'(\lambda)$$

formed by the ecliptic and the vertical. This is not quite true, but a convenient simplification since $v'(\lambda)$ is tabulated together with the zenith distances of points on the ecliptic [II, 13; Hei 1, 174] and thus easily found as a function of $\lambda(S^*)$ (see page 118). If we consider the small parallax triangle as plane we have accordingly

$$\pi_\lambda = S\Theta = \pi \cos v'(\lambda) \tag{7.25}$$

and

$$\pi_\beta = \Theta S' = \pi \sin v'(\lambda) \tag{7.26}$$

so that the parallax in longitude and latitude can be found from π by means of the table of chords.

This very simple method involves two approximations, both of them connected with the convenience of using the tables in II, 13.

First, the zenith distance z' of the star S' found from the tables is actually the zenith distance of S^*. In other words, the method presupposes that the apparent, geocentric latitude $\beta = S^*S'$ is zero.

Second, the angles v and $v'(\lambda)$ are considered as equal. This involves an error in v. This error disappears, however, if $\beta = 0°$.

According to Ptolemy already Hipparchus was aware of these inaccuracies and tried to correct them in a way which is described in the Almagest as *completely devoid of reason and contrary to all logic* – one of the few instances where Ptolemy is using hard words against his usually venerated predecessor.

In order to improve the theory Ptolemy devotes a long section [V, 19; Hei 1, 452–459] to a discussion of the cases in which the ecliptic is 1) perpendicular to the vertical, 2) parallel to it, and 3) forms an arbitrary angle with it, each time taking the latitude (of the Moon) into account. Here also there is a slight approximation since the calculation is founded on the assumption that

$$v = v'(\lambda) - \text{angle } S^*ZA \tag{7.27}$$

This is postulated with reference to Euclid I, 32, which means that the triangle AS^*Z is considered as plane. Yet for small latitudes this error is unimportant, so there is in fact an improvement. However, in the general case 3) the procedure of correcting the parallaxes is explained for the benefit of those who *might wish to take such small corrections into account*. Here we shall not further pursue this subject.

Eclipses of the Sun and the Moon

We have seen already that according to Ptolemy only the Moon exhibits a parallax great enough to influence observations of the accuracy available to him. This means that the theory cf parallax plays a particular role in Book VI, which is totally devoted to the theory of eclipses.

Apart from their astrological interest eclipses were important for purely scientific reasons as well. As we have seen, the theory of the motion of the Moon was founded upon observations of lunar eclipses; besides, already Hipparchus had called attention to their possible use in the determination of geographical longitudes[7]). This explains why Ptolemy devotes so much of the Almagest to the development of a theory of eclipses, immediately after the theories of the Sun and the Moon have been completed, although such a theory is not the main purpose of the Almagest.

It had been known for centuries before Ptolemy's time that eclipses happen only at the syzygies. A lunar eclipse takes place when the Moon enters the shadow of the Earth; it will then be opposite to the Sun with an elongation of 180° and appear as a Full Moon. On the other hand a solar eclipse occurs when the Moon intersects the line of sight from an observer (somewhere on the surface of the Earth) to the Sun; this may happen at New Moon with an elongation of 0°.

Since the shadow of the Earth is independent of the place of the observer, it follows that an eclipse of the Moon is an objective phenomenon in the sense that all observers will see its beginning (or maximum, or end) at exactly the same time. An eclipse of the Sun, however, on the contrary depends on the line of sight from the individual observer to the Sun; accordingly the geographical position of the observer plays a determining role, or – which is the same thing – the lunar parallax must be taken into account.[8])

It is now clear that there must be two main conditions for an eclipse.

1) The true elongation of the Moon from the Sun must be 0° or 180°. The times when that is the case can be determined by the theories of the longitudes of the Sun and Moon as set forth in Chapters 5 and 6 respectively. This first condition is necessary, but not sufficient since the Moon may be in true conjunction with the Sun without being near enough to the ecliptic to obscure the disc of the Sun; or it may be in true opposition without entering the shadow of the Earth.

2) We must, therefore, impose the supplementary condition that the latitude of the Moon is small enough, or, in other words that the Moon at conjunction or opposition is sufficiently near to one or another of the nodes. This means that the theory of the Moon's latitude is a prerequisite of the theory of eclipses.

7) The statement by Pliny (*Hist. nat.* II, 9, 54–58) that Hipparchus calculated eclipses for a period of 600 years after his own time has often been repeated (cf. Tannery 1893, p. 206). Considering the amount of work involved it is extremely unlikely that Hipparchus was able to accomplish such a task, even supposing that his lunar theory was sufficiently developed (see Toomer 1967, p. 146).
8) A very lucid explanation of how eclipses occur, and how the geographical position of the observer affects the size of a solar eclipse was given by Geminus (*Elem. Astron.*, X–XI).

The substance of Ptolemy's theory of eclipses is an attempt to express these two conditions in quantitative terms. How this is done will be very briefly examined in the following sections.

The Tables of Mean Conjunctions and Mean Oppositions

The first step is the construction of a set of tables for determining mean conjunctions and mean oppositions of the Sun and Moon [VI, 2; Hei 1, 462]. According to (5.2) and (6.2) we have for the mean longitudes at the arbitrary time t

$$\lambda_{\odot}(t) = \lambda_{m\odot}(t_0) + \omega_{\odot}(t - t_0) \tag{7.28}$$

and

$$\lambda_{m☽}(t) = \lambda_{m☽}(t_0) + \omega_t(t - t_0) \tag{7.29}$$

where t_0 is the standard epoch of Nabonassar (page 127) and

$$\begin{aligned} \lambda_{m\odot}(t_0) &= 330°;45 \ldots\ldots\ldots \text{(page 153)} \\ \lambda_{m☽}(t_0) &= 41°;22 \ldots\ldots\ldots \text{(page 182)} \\ \omega_{\odot} &= 0°;59,8,17 \ldots\ldots \text{(page 133)} \\ \omega_t &= 13°;10,34,58 \ldots \text{(page 164)} \end{aligned}$$

The condition that a mean conjunction takes place at the time t_n can be expressed as

$$\lambda_{m☽}(t_n) = \lambda_{m\odot}(t_n) + (n - 1) \cdot 360° \tag{7.30}$$

where n is an integer. Using (7.28) and (7.29) this condition can be written in the form

$$t_n = t_0 + \frac{\lambda_{m\odot}(t_0) - \lambda_{m☽}(t_0) - 360°}{\omega_t - \omega_{\odot}} + \frac{360°}{\omega_t - \omega_{\odot}} \cdot n \tag{7.31}$$

Here the denominator $(\omega_t - \omega_{\odot})$ is the mean, synodic, angular velocity of the Moon, i.e. the daily increase in mean elongation. Therefore the last term on the right hand side is $n \cdot T_s$, where $T_s = 29^d;31,50,8,20$ is a mean synodic month (see page 162). The second term is a constant which has the value $-5^d;47,33$, taking the parameters above. Thus (7.31) can be reduced to

$$t_n = t_0 - 5^d;47,33 + n \cdot 29^d;31,50 \ldots \tag{7.32}$$

This means that a mean conjunction took place (for $n = 0$) $5^d;47,33$ before the standard epoch, and that the subsequent ones occur with regular intervals of 1 mean synodic month.

In the same way we can state the condition that a mean opposition takes place at the time T_n

$$\lambda_{m☽}(T_n) = \lambda_{m\odot}(T_n) + (n - \tfrac{3}{2}) \cdot 360° \tag{7.33}$$

which leads to the result

$$T_n = t_0 + \frac{\lambda_{m\odot}(t_0) - \lambda_{m☽}(t_0) - 180^\circ}{\omega_t - \omega_\odot} + \frac{360^\circ}{\omega_t - \omega_\odot} \cdot (n - 1) \qquad (7.34)$$

or numerically

$$T_n = t_0 + 9^d;58,22 + (n - 1) \cdot 29^d;31,50 \ldots \qquad (7.35)$$

Thus a mean opposition took place $9^d;58,22$ after the standard epoch ($n = 1$) in (7.35), and the subsequent ones happen with successive intervals of 1 mean synodic month.

The times of mean conjunctions t_n and of mean oppositions T_n are now tabulated in two separate tables. The *Table of (Mean) Conjunctions* [Table 1, VI, 3; Hei 1, 446] is constructed as follows:

Column I contains the number of years in 25 year periods after t_0, i.e. the years Nabonassar 1, 26, 51 ...

Column II gives for each of the years in Column I the date of the first mean conjunction (mean New Moon) which always occurs within the first month (Thoth) of the year. The numbers in this column vary very slowly due to the fact that 25 Egyptian years are very nearly equal to 309 mean synodic months. This is, perhaps, the reason why Ptolemy uses 25 year intervals here, as later everywhere in the *Handy Tables* (see page 134), while he uses 18 year intervals elsewhere in the Almagest, for instance in the tables of mean motion.

Column III gives the distance of the Mean Sun (and also the Moon) from the solar apogee at the time of the mean conjunction, i.e. the function

$$a_{m\odot}(t_n) = \lambda_{m\odot}(t_n) - \lambda_{a\odot} \qquad (7.36)$$

called the mean argument of the Sun, cf. (5.17).

Column IV lists the mean argument $a_m(t_n)$ of the Moon at the time of the conjunction.

Column V is the argument of latitude of the Moon – 90°, i.e. the distance

$$\lambda_{m☽}(t_n) - \lambda_n(t_n) - 90^\circ$$

of the mean Moon from the highest point of the inclined circle, cf. (6.67–68).

The *Table of (Mean) Oppositions* (mean Full Moons) [Table 2; VI, 3; Hei 1, 468] has exactly the same structure. Because of the long interval of 25 years these two tables have to be supplemented by two other ones, making it possible to determine New and Full Moons inside a 25 year period [Table 3, Hei 1, 470], and inside the singular years [Table 4, Hei 1, 471] by adding a number of days taken from table 3 or 4 to the dates t_n or T_n found in table 1 or 2. In all these tables a day means a mean solar day; consequently it can be necessary to introduce a correction in order to get the conjunction or opposition times in civil time, cf. page 157.

Determination of the True Syzygies

Until now we have seen how mean conjunctions and oppositions can be found. But eclipses depend on **true conjunctions** or **true oppositions**. Therefore, the next step must be to find the true syzygies from the mean ones. Suppose that we have found that a mean conjunction will occur at the time t_n at the mean ecliptic longitude $\lambda_{m\odot}(t_n) = \lambda_{m☽}(t_n)$. According to (5.12) the true longitude of the Sun at the same moment will be

$$\lambda_{\odot}(t_n) = \lambda m_{\odot}(t_n) + q(a_m) \tag{7.37}$$

where $q(a_m)$ is found in the *prosthaphairesis* table of the Sun by means of the argument (7.36) taken from Column III of the Table of Conjunctions. Similarly, the true longitude of the Moon is given by

$$\lambda_{☽}(t_n) = \lambda_{m☽}(t_n) + p(c, a_v) \tag{7.38}$$

where the lunar *prosthaphairesis* $p(c, a_v)$ is found as explained on page 198.

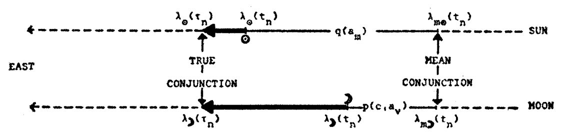


Fig. 7.7

If it so happens that the true longitudes are equal, then the first condition for an eclipse is satisfied. In general this will not be the case. Let us suppose, first, that (cf. Figure 7.7).

$$\lambda_{\odot}(t_n) > \lambda(t_n)$$

This means that at the time t_n of the mean conjunction the true Moon is behind the Sun and the true conjunction is delayed a certain time Δt_n. The interval can be determined approximately by the relation

$$\lambda_{\odot}(t_n) + \dot{\lambda}_{\odot} \cdot \Delta t_n = \lambda_{☽}(t_n) + \dot{\lambda}_{☽} \cdot \Delta t_n \tag{7.39}$$

where the true angular velocities $\dot{\lambda}_{\odot}$ and $\dot{\lambda}_{☽}$ are supposed to remain constant during the time Δt_n. Solving with respect to Δt_n we get the additional time

$$\Delta t_n = \frac{\lambda_{\odot}(t_n) - \lambda_{☽}(t_n)}{\dot{\lambda}_{☽} - \dot{\lambda}_{\odot}} \tag{7.40}$$

Thus the time of the true conjunction is

$$\tau_n = t_n + \Delta t_n = t_n + \frac{\lambda_{\odot}(t_n) - \lambda_{☽}(t_n)}{\dot{\lambda}_{☽} - \dot{\lambda}_{\odot}} \tag{7.41}$$

From (7.40) we find the additional motion of the Moon from mean to true conjunction as

$$\Delta\lambda_{☽}(t_n) = \Delta t_n \cdot \dot{\lambda}_{☽} = [\lambda_{\odot}(t_n) - \lambda_{☽}(t_n)] \cdot \frac{\dot{\lambda}_{☽}}{\dot{\lambda}_{☽} - \dot{\lambda}_{\odot}} \tag{7.42}$$

so that the place of the true conjunction is

$$\lambda(\tau_n) = \lambda_{☽}(t_n) + \Delta\lambda_{☽}(t_n) = \lambda_{☽}(t_n) + [\lambda_{\odot}(t_n) - \lambda_{☽}(t_n)] \cdot \frac{\dot{\lambda}_{☽}}{\dot{\lambda}_{☽} - \dot{\lambda}_{\odot}} \tag{7.43}$$

Ptolemy's Rules of Thumb

The formulae (7.41) and (7.43) solve the problem of determining true conjunctions in an approximative way. However, they are not well adopted to practical calculations since they presuppose that we know the angular velocities $\dot{\lambda}_{\odot}(t_n)$ and $\dot{\lambda}_{☽}(t_n)$ of the Sun and Moon at the time of the mean conjunction; but in the Almagest only longitudes are determined as explicit functions of time; angular velocities are not. Now instead of (7.41) and (7.43) Ptolemy uses two approximative procedures, or rules of thumb, which are not quite obvious and need some comments.

Beginning with the place of the true conjunction, Ptolemy declares that it can be found as the true longitude at the mean conjunction added to the true elongation at the same time augmented by $\frac{1}{12}$ [VI, 4; Hei 1, 474]. In other words (7.43) is replaced by

$$\lambda(\tau_n) = \lambda_{☽}(t_n) + [\lambda_{\odot}(t_n) - \lambda_{☽}(t_n)] \cdot (1 + \tfrac{1}{12})$$

where the last term

$$\Delta\lambda_{☽}(t_n) = [\lambda_{\odot}(t_n) - \lambda_{☽}(t_n)] \cdot (1 + \tfrac{1}{12}) \tag{7.45}$$

is the additional motion of the Moon from mean to true conjunction according to Ptolemy.

Later this rule is made plausible by a numerical example [VI, 5; Hei 1, 482] in which the true elongation at mean conjunction is 7°;24. In order to overtake the Sun the Moon must first move through these 7°;24 towards the East. During this motion the Sun will move approximately $\frac{1}{13}$ of this distance, or 0°;34, further towards the East. While the Moon moves this distance the Sun will move through $\frac{1}{13}$ of it, or 0°;3, of which another $\frac{1}{13}$th will be of no importance. Thus the additional motion of the Moon is

$$0°;34 + 0°;3 = 0°;37 = 7°;24 \cdot \tfrac{1}{12}$$

This example implies a method in which the longitude of the true conjunction is found as $\lambda_{☽}(t_n)$ plus the sum of an infinite series

$$7°;24 + 7°;24 \cdot \tfrac{1}{13} + 7°;24 \cdot (\tfrac{1}{13})^2 + \dots$$

approximated by the first three terms. This is one of the rare instances (perhaps the only one) where Ptolemy makes use of convergent series.

To compare Ptolemy's rule with (7.43) we write the latter relation in the form

$$\lambda(\tau_n) = \lambda_☽(t_n) + [\lambda_\odot - \lambda_☽] + [\lambda_\odot - \lambda_☽] \cdot \frac{\dot{\lambda}_\odot}{\dot{\lambda}_☽ - \dot{\lambda}_\odot} \tag{7.46}$$

Here the second factor of the last term depends on the true angular velocities, and is thus a function of time, the value of which at the time t_n Ptolemy is unable to determine. But if we replace $\dot{\lambda}_\odot$ and $\dot{\lambda}_☽$ with their mean values $\omega_\odot$ and ω_t we have (see page 133 and 164)

$$\frac{\dot{\lambda}_\odot}{\dot{\lambda}_☽ - \dot{\lambda}_\odot} \approx \frac{\omega_\odot}{\omega_t - \omega_\odot} = \frac{0°;59{,}8}{12°;11{,}27} = \frac{1}{12;22} \tag{7.47}$$

This shows that even with the mean angular velocities Ptolemy's rule (7.43) is not quite correct since it implies that

$$\frac{\dot{\lambda}_\odot}{\dot{\lambda}_☽ - \dot{\lambda}_\odot} = \tfrac{1}{12} \tag{7.48}$$

Thus (7.44) rests on a double approximation: First the true angular velocities are replaced by their mean values, and next the fraction $\frac{1}{12}$ in (7.48) is substituted for the more exact value (7.47).

The time τ_n of the true conjunction is found by another rule of thumb [VI, 4; Hei 1, 474] which states that the additional time (cf. 7.40) is the additional motion (7.45) divided by the true angular velocity $\dot{\lambda}_☽(t_n)$ of the Moon at the time of the mean conjunction. This rule follows from (7.42) written in the form

$$\Delta t_n = \frac{\Delta\lambda_☽(t_n)}{\dot{\lambda}_☽} \tag{7.49}$$

This second rule has a particular interest since it presupposes that we know the true, time-dependent angular velocity $\dot{\lambda}_☽$ at the time of the mean conjunction. Thus Ptolemy has to face the problem of determining angular velocities as functions of time.

The Instantaneous, Angular Velocity of the Moon

This is done in a rather obscure passage [VI, 4; Hei 1, 474f.] in which Ptolemy outlines the following procedure for finding $\dot{\lambda}_☽(t_n)$:

1) Find (from (6.45)) the true argument of the Moon $a_v(t_n)$ at the time of the mean conjunction.
2) Enter with a_v in Column IV of the general table of equations of the Moon and find (see page 196) $p_1(a_v) = p(0°;a_v)$, cf. (6.53).
3) Enter in the line above and find $p_1(a_v - d)$, where d is 6° or 3° according to whether we are in the upper or lower part of the table.

4) Calculate $\dot{\lambda}_{☽}(t_n)$ as

$$\dot{\lambda}_{☽}(t_n) = \omega_t + \frac{p_1(a_v) - p_1(a_v - d)}{d} \cdot \omega_a \tag{7.50}$$

where ω_a is the (hourly) mean anomalistic angular velocity, and ω_t the mean tropical angular velocity.

To understand this procedure we differentiate (7.38) with respect to time:

$$\dot{\lambda}_{☽} = \dot{\lambda}_{m☽} + \frac{dp(c, a_v)}{dt} = \omega_t + \frac{\partial p}{\partial c} \cdot \dot{c} + \frac{\partial p}{\partial a_v} \cdot \dot{a}_v \tag{7.51}$$

The question is now whether this relation is equivalent to (7.50). Now the first term ω_t is the same in both equations. The second term in (7.50) is equivalent to the third term in (7.51) if we have

$$\frac{p_1(a_v) - p_1(a_v - d)}{d} = \left.\frac{\partial p}{\partial a_v}\right|_{c = 0°} \tag{7.52}$$

and

$$\dot{a}_v = \omega_a \tag{7.53}$$

Finally the middle term in (7.51) has no counterpart in (7.50).

First, we notice that (7.52) is approximately true according to the definition of a partial derivative; the latter was, of course, unknown to Ptolemy who had no clear idea of the concept of limits; his handling of an infinite series (page 224) in no way contradicts this fact.

Next, the condition $c = 0°$ is only fulfilled at the mean syzygies (see 6.41). But since eclipses occur near these points, we have to admit that (7.52) is valid as far as the theory of eclipses is concerned.

The relation (7.53) is more problematic. Differentiating (6.45) we find

$$\dot{a}_v = \omega_a + \frac{dq}{dt} = \omega_a + \frac{dq}{dc} \cdot \dot{c} \tag{7.54}$$

which shows that (7.53) is admissible only if $dq/dt = 0$, i.e. if q is a constant the value of which must be zero, since we have $q = 0°$ at both the apogee and the perigee of the deferent (cf. 6.47). This is possible only if the lunar orbit has the eccentricity $e = 0$. But in that case it follows from (6.46) that $\rho = R$ is a constant, and from (6.49) that $p(c, a_v)$ is independent of c, so that the middle term of (7.51) must vanish.

The conclusion is that Ptolemy's method of computing the actual angular velocity $\dot{\lambda}_{☽}$ of the Moon at a given time is understandable only in terms of the first, or concentric lunar model (page 167). This is, perhaps, an indication that the method is due to Hipparchus, who created the First Model, and that Ptolemy did not consider it sufficiently important to work out a similar method based on his own lunar models.

The Limits of Eclipses

We have now seen how effectively Ptolemy dealt with the first condition that an eclipse may occur (cf. page 220) in so far as he has developed (approximative) methods for determining the time and place of true conjunctions and oppositions. The second condition was that the latitude of the Moon is small enough to bring it into contact with the apparent disc of the Sun, or with the shadow of the Earth. Therefore the next step must be to investigate how near to one or another of the nodes the true conjunction or opposition must take place in order to make the contact possible. This is the problem of the limits of solar and lunar eclipses which is dealt with in a separate chapter [VI, 5; Hei 1, 476].

The calculation of these limits rests on the known values of the apparent radii of the Moon, the Sun, and the Earth's shadow. Now Ptolemy had found the former to be 0°;31,20 by means of two old lunar eclipses in both of which the Moon was near to the apogee of the epicycle, i.e. at its maximum distance from the Earth (page 209). In other words, the result was a minimum value of the apparent diameter. The maximum value is now found by a similar calculation based on two other lunar eclipses in which the Moon was at the perigee of the epicycle. The first (appendix A, N° 33) was observed in 174 B.C. by an unknown Alexandrian astronomer, the second (N° 42) in 141 B.C. by Hipparchus at his Rhodes observatory. For the sake of brevity we omit the details of the observations and the following computations. The result is that the maximum apparent diameter of the Moon is 0°;35,20, and that here the shadow of the Earth has a radius of 0°;46; this is still very nearly $2\frac{3}{5}$ times the apparent radius of the Moon [VI, 5; Hei 1, 479–480, cf. (7.6)].

The general problem of limits is the following: What is the maximum distance of the centre of the Moon from one of the nodes compatible with a contact between the Moon on the one hand, and the Sun or the shadow of the Earth on the other? If there were no parallactic effects then this question could be answered very easily.

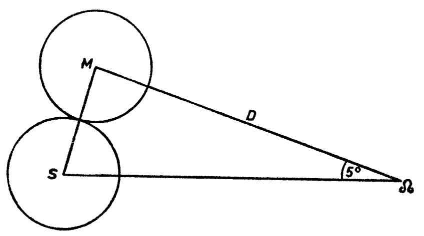


Fig. 7.8

In Figure 7.8 the Moon is just touching the Sun, so that the distance SM is equal to the sum of the apparent radii of the two discs. Since we get the largest limits with the greatest apparent size of the Moon, we must put

$$SM = 0°;15,40 + 0°;17,40 = 0°;33,20$$

and calculate the limit of a solar eclipse as

$$☊M = 0°;33,20 \cdot \cot 5° \approx 6°$$

Replacing the radius of the Sun with that of the shadow of the Earth we could determine the limit of a lunar eclipse in precisely the same way. However, this simple solution has to be modified in the case of a solar eclipse because of the effect of both the solar and the lunar parallax. Ptolemy effects this modification in a rather crude way which is neither elegant nor satisfactory, involving too many uncontrolled approximations. Therefore, in the following section we shall consider only the general lines of Ptolemy's solution, condensing his long arguments as much as possible without examining how good the various approximations are.

The Influence of Parallax

He begins by considering a situation in which the Sun and Moon are in what one could call an apparent conjunction. What this means appears from Figure 7.9 which

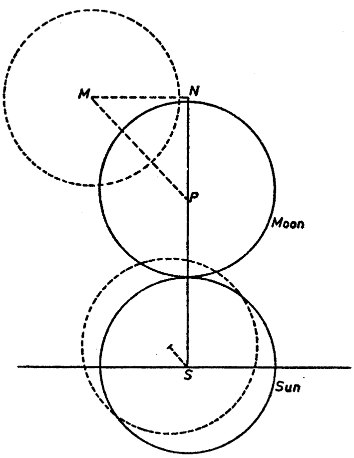


Fig. 7.9

shows the geocentric positions of the two discs as dotted circles with the Moon North of the ecliptic. Due to parallax, the apparent Sun has a slightly greater zenith distance, while the zenith distance of the Moon may be increased much more. The difference between the lunar and solar parallax in altitude is denoted by $MP = \Pi$ and split into a component $MN = \Pi_\lambda$ along the ecliptic and another $NP = \Pi_\beta$ perpendicular to it. Due to these components, the apparent disc of the Moon is now just touching that of the Sun in such a way that the centres of the two luminaries have the same ecliptic longitude. We can now express the distance SN as

$$SN = SP + PN = 0°;33,20 + \Pi_\beta$$

The distance of N from the nearest node is found by taking the inclination into account as

$$☊N = SN : \sin 5^\circ$$

By the new approximation of considering NM as a straight line through the node we can express the distance of M from the node as

$$☊M = ☊N + NM = \frac{0°;33,20 + \Pi_\beta}{\sin 5^\circ} + \Pi_\lambda \tag{7.55}$$

By this amount at most the position of the true Moon, measured along the inclined circle, may differ from the nearest node to make a solar eclipse just possible.

Now it will be more convenient to know the corresponding maximum distance of the mean Moon, since it is easier to calculate mean positions than true positions. According to (5.25) the maximum distance between the mean and true Sun is $q_{max} = 2°;23$, while the corresponding value for the Moon at the syzygies is 5°;1 (page 180). Therefore the maximum mean elongation, that is the maximum distance between the mean Moon and the mean Sun, may amount to as much as $2°;23 + 5°;1 = 7°;24$ at a true conjunction. But this is the same as the maximum distance between the true Moon and the true Sun at a mean conjunction. Supposing the Moon to be behind the Sun we already know (page 224) that the latter will move 0°;37 while the Moon overtakes it. Thus a solar eclipse is possible even if at the true conjunction the true Sun is $2°;23 + 0°;37 = 3°$ out of place relative to the mean Sun at the mean conjunction. Reckoning these 3° as an arc on the inclined circle we must add them to (7.55) in order to get the absolute limit

$$L_\odot = \frac{0°;33,20 + \Pi_\beta}{\sin 5^\circ} + \Pi_\lambda + 3^\circ \tag{7.56}$$

of an eclipse of the Sun. A similar formula can be derived for positions of the Moon South of the ecliptic.

In (7.56) the relative parallaxes Π_λ and Π_β of the Sun and the Moon will depend on the zenith distances of the two bodies and, therefore, on the geographical latitude φ of the observer (cf. 7.12–13). Thus the limit $L_\odot$ will depend on φ. In order to evaluate this effect at different places on the Earth Ptolemy chooses what he considers to be two extreme localities of the inhabitable world. The one is the mouth of the Borysthenes (Dnieper) in the North, the other is the Nile island of Meroë in the South [VI, 5; Hei 1, 481]. Neglecting the parallax of the Sun, we have here the following data (cf. page 108).

	Borysthenes	Meroë
Longest day	16^h	13^h
Geogra. latitude	48°;32	16°;27
Maximum of Π_λ	0°;15	0°;30
Maximum of Π_β	0°;58	0°;8
Limit $L_\odot$	20°;41	11°;22

At places with latitudes between 48° and 16° the limits will lie between the two extreme values.

The limits of a lunar eclipse are much easier to calculate since there are no complicating parallactic effects. The maximum radius of the Moon was found to be 0°;17,40 (page 227) and the corresponding radius of the shadow of the Earth 0°;45,56. Their sum is 1°;3,36.

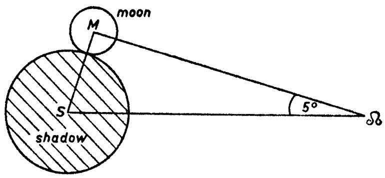


Fig. 7.10

The hypotenuse of the triangle in Figure 7.10 is

$$1°;3,36 : \sin 5° = 12°;12$$

to which we have to add 3° as before. This gives the maximum distance from the nodes

$$L_{☽} = 15°;12 \tag{7.57}$$

of the mean Moon at mean conjunction compatible with contact between the Moon and the shadow.

The Frequency of Eclipses

How soon after one eclipse can another of the same kind take place? The answer to this question is of some importance for the prediction of future eclipses, since it will enable us to rule out a number of conjunctions and oppositions as unproductive of eclipses, and thus to save unnecessary calculations. Ptolemy therefore devotes a long chapter [VI, 6; Hei 1, 485] to the question of the frequency of eclipses, or rather to the possible time intervals between two consecutive eclipses of the same kind. We shall not follow this part of the theory of eclipses in any detail, but only briefly indicate the general trend of the argument.

In Figure 7.11 a the limits of a lunar eclipse are marked on the inclined circle by the shaded arcs extending 15°;12 to each side of each of the nodes. The anecliptic intervals between these arcs are 149°;36. Now suppose that at a given time we have an eclipse of the Moon. After the lapse of an integer number n of synodic months the Sun and the Moon will again be in opposition; but then an eclipse will occur only if the Moon's motion in latitude has again brought it inside one of the shaded arcs in Figure 7.11 a.

If both the Sun and the Moon moved with uniform velocities we would find

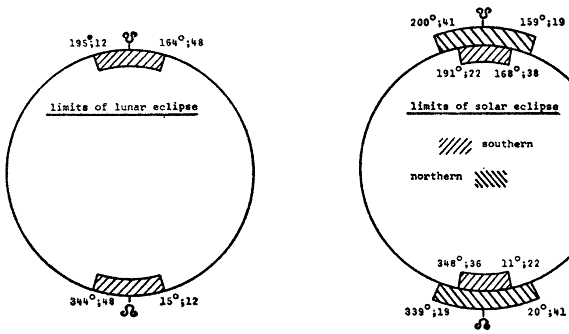


Fig. 7.11 a–b

from the tables of mean motion [IV, 3; Hei 1, 282] that in $n = 6$ synodic mean months, or $6 \cdot 29^d;31,50 = 177^d;11$ the mean motion in latitude is $184°;1 + 360°$. This means that the 6th opposition would take place inside the shaded arc around the other node (except in a few rare cases) so that a new eclipse would be produced. But we know that both the Sun and the Moon move with varying velocities so that the various true synodic months are of unequal length. Taking this into account, and considering also the effect of parallax, Ptolemy succeeds in formulating a number of rules which can serve as exclusion principles in the calculations of eclipses. They can be summarized as follows:

The interval between two consecutive lunar eclipses may be 5 months, but never 7 months.

The interval between two consecutive solar eclipses is never 1 month. It may be 5 months or 7 months.

It is worth noticing that here Ptolemy omits any reference to the Saros period of 223 synodic months $= 18^a 11^d$, after which eclipses will, in general, repeat themselves (see page 162).

The Magnitude of a Solar Eclipse

Let us consider an eclipse of the Sun, assuming that we have already found the parameters of an apparent conjunction of the Sun and the Moon by the methods sketched above, with due respect to both parallax and prosthaphairesis of both the Sun and Moon. We can then ask the question how great the eclipse will be – total or partial, and – in the latter case – how much of the Sun will be eclipsed by the Moon.

As already mentioned (page 171) the magnitude d of an eclipse is measured in digits (one d i g i t being the twelfth part of the diameter of the Sun) and expressed as the number of digits covered by the Moon when the eclipse is at its greatest, i.e. precisely at the computed time of the apparent conjunction.

This number d must depend mainly on the angular distance between the centres of the Sun and the Moon, which is a function of the distance D of the lunar centre from the node. But to a lesser degree the number of digits must depend also on the

distance Δ of the Moon from the Earth expressed by (6.51). Thus the magnitude $d = d(D, \Delta)$ is a function of a strong variable D and a weak variable Δ. This is a type of function frequently used in the Almagest. Ptolemy handles it according to his standard method; he computes d as a function of the strong variable D alone, for two fixed values of the weak variable Δ, using an interpolation method to account for the variation of the latter.

First let us assume that Δ has its maximum value $\Delta_1 = (R + e) + r$ (cf. page 215), so that the apparent radius of the Moon is minimal and equal to that of the Sun, or 0°;15,40. Figure 7.8 shows a situation where $d = 0$ digits. Here

$$MS = 0°;31{,}20$$

and

$$D = 0°;31{,}20 \cdot \cot 5° \approx 6°$$

For an eclipse of magnitude $d = 1$ the distance SM will be $\frac{11}{12}$ of the former value, and the whole triangle – considered as plane – will be reduced in the ratio 11 : 12. Accordingly we have D = 5°;30. In this way we arrive at the simple relation

$$D = 6°\left(1 - \frac{d}{12}\right) \tag{7.58}$$

which is tabulated in the first *Table of Solar Eclipses* [VI, 8; Hei 1, 519].

The Tables of Solar Eclipses

Here Column III contains the integer values of d from 0 to 12 and back to 0 again. Column I does not contain the argument D, but (90° – D), varying from 84° to 96°. This only means that the argument of the table is reckoned from the northern top of the inclined circle instead of from the ascending node. In other words, it is the usual argument of latitude (cf. page 201). Column II contains 360° – (90° – D) = 270° + D, corresponding to the same number of digits as the argument in Column I.

Now there remains only Column IV of the eclipse table, containing the motion of the Moon relative to the Sun from the first contact to the central (maximum) position. In Figure 7.12 the first and last contacts are at F and L, corresponding to the positions M_1 and M_2 of the centre of the Moon. In the central position the centre is at M, and

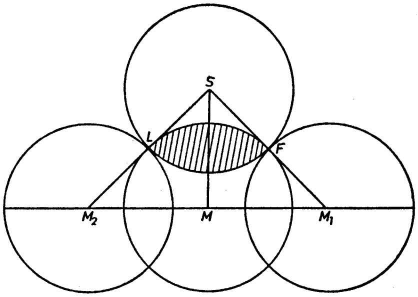


Fig. 7.12

the problem is to determine M_1M (equal to MM_2) under the assumption that the figure can be regarded as plane.

Here M_1S is the sum of the radii $= 0°;31,20$. The distance MS is equal to this sum minus the obscured part of the diameter of the Sun, or

$$\begin{aligned} MS &= 0°;31,20 - 0°;31,20 \cdot \frac{d}{12} \\ &= 0°;31,20 \cdot \left(1 - \frac{d}{12}\right) \end{aligned}$$

where d is the magnitude of the eclipse in digits. By the theorem of Pythagoras we have

$$M_1M = \left[0°;31,20^2 - 0°;31,20^2\left(1 - \frac{d}{12}\right)^2\right]^{\frac{1}{2}}$$

which can be contracted to

$$M_1M = \tfrac{1}{12} \cdot 0°;31,20 \cdot \sqrt{d(24 - d)} \qquad (7.59)$$

This distance is listed in minutes and seconds of arc as a function of d in Column IV. It varies from 0°;0 for $d = 0$ to 0°;31,20 at a total eclipse ($d = 12$). As explained later on [VI, 10; Hei 1, 532] half the duration of the eclipse is found by dividing $M_1M \cdot (1 + \frac{1}{12})$ by the instantaneous velocity of the Moon; the fraction $\frac{1}{12}$ accounts for the extra motion of the Sun (cf. page 224). In Mediaeval versions of the eclipse tables the fourth column is headed *minuta casus*. Also the time of the first contact can now be determined by subtracting half the duration from the time of the central phase (the apparent conjunction). Similarly the time of the last contact is found by addition of the two times.

The second table of solar eclipses is computed under the assumption that the distance Δ of the Moon from the Earth is minimal, and that consequently the apparent diameter of the Moon has its maximum value 0°;35,20. The method is very much the same as before. But since the Moon now appears greater than the Sun there is a slight difference around the central phase of the eclipse. In the former case a total eclipse had a sharply defined moment of totality, but here there will be a short interval of time during which the Sun is totally obscured. Actually the Moon is able to cover an arc of 0°;35,20 which, measured by the digits of the Sun is

$$0°;35,20 : (0°;31,20 : 12) = 13.53 \text{ digits}$$

instead of 12 digits. Half of the difference is 0.77 digits, which Ptolemy rounds off to $\frac{4}{5}$ digits. Therefore Column III of the second table has a maximum of $12\frac{4}{5}$ digits for an argument of latitude of 90° or 270°.

The Interpolation Table

With these two tables Ptolemy has described the magnitude and half duration of a solar eclipse as a tabulated function of the strong variable D, but only for two extreme

values $\Delta_1 = (R + e) + r$ and $\Delta_2 = (R + e) - r$ of the weak variable (cf. page 86). In general the distance Δ of the Moon from the Earth depends on both the true argument a_v and the centrum c of the Moon, i.e. of its position on the epicycle and the position of the latter on the deferent. Now all eclipses take place when the epicycle is near the apogee of the deferent (page 215) where $c = 0°$. When we now wish to compute an eclipse for values of Δ between Δ_1 and Δ_2 we can disregard the centrum of the Moon and need only take the argument into account.

This is done in the following way: When D has been determined as the Moon's distance from the node at the time of the apparent conjunction we can, for each value of D, find $d(D, \Delta_1)$ and $d(D, \Delta_2)$ from the first and second eclipse tables. This implies, in general, some linear interpolation, since D will but rarely coincide with one of the tabulated values corresponding to integer values of d.

The next step is to find the true argument a_v of the Moon at the same time by the procedure given in the lunar theory (page 195).

Finally the true value of $d = (D, \Delta)$ is calculated by means of a procedure corresponding to the interpolation formula

$$d(D, \Delta) = d(D, \Delta_1) + [d(D, \Delta_2) - d(D, \Delta_1)] \cdot \frac{f'(a_v)}{60} \tag{7.60}$$

Here $[d(D, \Delta_2) - d(D, \Delta_1)]$ is the difference of the values taken from the two tables for the argument D, while $f'(a_v)$ is an interpolation function already used in the theory of parallaxes; it is defined by (7.16) and tabulated in Column VII of the general parallax table [V, 18; Hei 1, 442]. This column contains $f'(a_v)$ tabulated at intervals of a_v of 2° from 0° to 90°, while in (7.60) a_v varies from 0° to 360°, corresponding to a complete revolution on the epicycle. Ptolemy changes Column VII of the parallax table into a small table of corrections adapted to (7.60). This table [VI, 8; Hei 1, 522] gives $f'(a_v)$ for all values of a_v from 0° to 360° at intervals of 6°.

One more table of eclipses is found in this chapter [ibid.], namely a *Table of the Eclipsed Area* of the Sun as function of the number of digits $d(D, \Delta)$. This table raises no particular problems. But it is interesting to notice that it is computed by means of a value of π equal to $3\frac{17}{120}$. This is approximately, but not exactly, the mean value between $3\frac{1}{7}$ and $3\frac{10}{71}$, i.e. the two limits of the interval inside which π was proved to lie by Archimedes, to whom Ptolemy refers (cf. Heath, *Archimedes*, p. 93f).

Lunar Eclipses

The computation of lunar eclipses [VI, 9; Hei 1, 523] follows essentially the same scheme as that of eclipses of the Sun, except that the disc of the Sun must be replaced by the circular cross section of the shadow of the Earth. Since the shadow was supposed to be $2\frac{3}{5}$ times as large as the Moon, both at the longest and the shortest distances, the Moon can be completely immersed (totally eclipsed) for a considerable time. This can

be described by saying that the number of digits can be considerably larger than 12. In a total eclipse of the Moon we must, therefore, distinguish between [cf. VI, 11; Hei 1, 537]

1) The first contact (M_1)
2) The beginning of the totality (M_2)
3) The central phase (M)
4) The end of the totality (M_3)
5) The last contact (M_4)

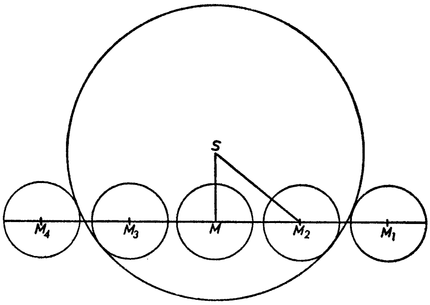


Fig. 7.13

These principal situations are shown in Figure 7.13. The distances of M_1 and M_2 from the centre S of the shadow can be found geometrically as before, and the same is the case with the motion of the Moon between the principal situations.

The results are tabulated in two tables, one for the maximum distance Δ_1, another for the minimum distance Δ_2 [VI, 8; Hei 1, 520]. In each table the first 3 columns are similar to those of the tables of solar eclipses; in Column III we notice that the number of digits can amount to 21 for a central eclipse.

In Column IV we find the motion of the Moon from first contact to complete immersion expressed in minutes and seconds of arc. Divided by the instantaneous velocity of the Moon and multiplied by $(1 + \frac{1}{12})$ this number gives the duration of the immersion, i.e. the time from the first contact to the beginning of totality. In Latin astronomy this was called the *minuta casus* of the eclipse.

In these lunar tables a fifth column is added, containing the motion of the Moon from the time of complete immersion to the middle of the total phase, from which half of the duration of the total phase can be found, that is half of the time of the Moon's stay inside the shadow. This was called in Latin *minuta dimidie more.*

To his theory of eclipses Ptolemy further adds a section on the position angle of the principal phases of eclipses [VI, 11; Hei 1, 535], that is the angle between the ecliptic and the line connecting the centres of the two discs. The results are given as functions of the number of digits in a separate table [VI, 12; Hei 1, 544]. Finally it is shown how this table can be used for finding the angle between the central line and the horizon [VI, 13; Hei 1, 545].

CHAPTER 8

The Fixed Stars

Introduction – the Fundamental Stars

In Books VII and VIII of the Almagest Ptolemy interrupts the development of the planetary theories in order to deal with a number of problems connected with the fixed stars. The reason for this is explained in the prologue to this second part of the Almagest [VII, 1; Hei 2, 2], where Ptolemy tells his friend or protector Syrus (see page 26) that the further development of planetary theory presupposes a chapter on *the so-called fixed stars*. The reason is that while the solar theory was founded upon simple observations of solstitial and equinoctial times, and the lunar theory upon eclipses, the theories of the remaining five planets are, to a great extent, founded upon exact determinations of planetary longitudes derived from the distance of the planet in question from a fixed star, measured with the *astrolabon* (see page 183). Thus a position of Mercury is referred to Aldebaran (α Tauri) [IX, 7; Hei 2, 262] or to Regulus (α Leonis) [ibid.; Hei 2, 263] just as a longitude of Venus is found relative to that of Antares (α Scorpii) [X, 3; Hei 2, 303].

Thus the planetary theories presuppose a knowledge of the exact coordinates of a number of fundamental reference stars, among which the most important are listed in the table below. The names of the stars in Column I are modern. In Column II the numbers refer to the observations (listed in App. A) in which the star in question is used as a reference star.

Table of Reference Stars

I	II
α Tauri	60, 67, 71, 81, 87, 88, 90
β Tauri	23
β Aurigae	23
α Geminorum	25
β Geminorum	25
α Leonis	58, 68, 79, 84
γ Virginis	29
α Virginis	24, 80, 86
α Librae	28
β Scorpii	17, 20, 26, 80

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_8,

α Scorpii	66, 94
α Capricorni	76
β Capricorni	76
δ Capricorni	22

All these stars are of the first, second or third order of magnitude. The rest are fainter than 3^m, viz. α and 38 Arietis, ξ Tauri, ξ Geminorum, δ Cancri, η Virginis, δ Scorpii, ν Capricorni, and φ Aquarii. It is seen that they are all situated near to the ecliptic (except β Aurigae) and distributed more or less evenly in the zodiacal belt in which all the apparent paths of the planets are found. Yet the four fundamental stars most often referred to (α Tauri, α Leonis, α Virginis, and β Scorpii) are all found on one half of the ecliptic.

The problem is now if it is possible to determine the coordinates of the reference stars, not only at the time when Ptolemy made his own observations, but also at the much earlier times of the ancient observations to which he so often refers. This problem has several aspects, investigated in the first chapter of Book VII [VII, 1; Hei 2, 2], where Ptolemy tries to establish two fundamental properties of the stars of the firmament.

1) The fixed stars are fixed only in so far as they have no proper motion relative to each other, and

2) they share a common, regular and uniform motion towards the East relative to the vernal equinox.

The Absence of Proper Motions

The former statement is not proved by direct measurements of relative stellar distances by means of instruments with graduated circles. It is true that with his *astrolabon* Ptolemy had an instrument which could be easily adapted to this purpose as later observations show [e.g. IX, 7; Hei 2, 262]. But this was, perhaps, the first instrument of its kind in the history of astronomy, and the distances measured with it had no counterparts in earlier times with which they could be compared. Therefore Ptolemy had to rely upon the extremely crude but beautifully simple method of alignments, which dates from a very early stage of Egyptian astronomy[1]). It makes use of no other equipment than a thin string held taut before the eyes of the observer, who in this way is able to decide whether or not three or more stars appear on the same straight line. (cf. Figure 8.1).

1) Egyptian astronomers and surveyers had a particular instrument consisting of a slit palm-leaf (merkhet) and plumb-line for observing the passage of a star across the meridian (see the illustration in L. Borchardt, *Altägyptische Zeitmessung*, Pl. XVI, Berlin, 1920). This may have accustomed them to connect the stars by a string held in front of the eyes. It is interesting to notice that the method of alignment was used as late as A.D. 1604 when Galileo proved that the nova of that year was a fixed star (see the *Opera Omnia* II, p. 279).

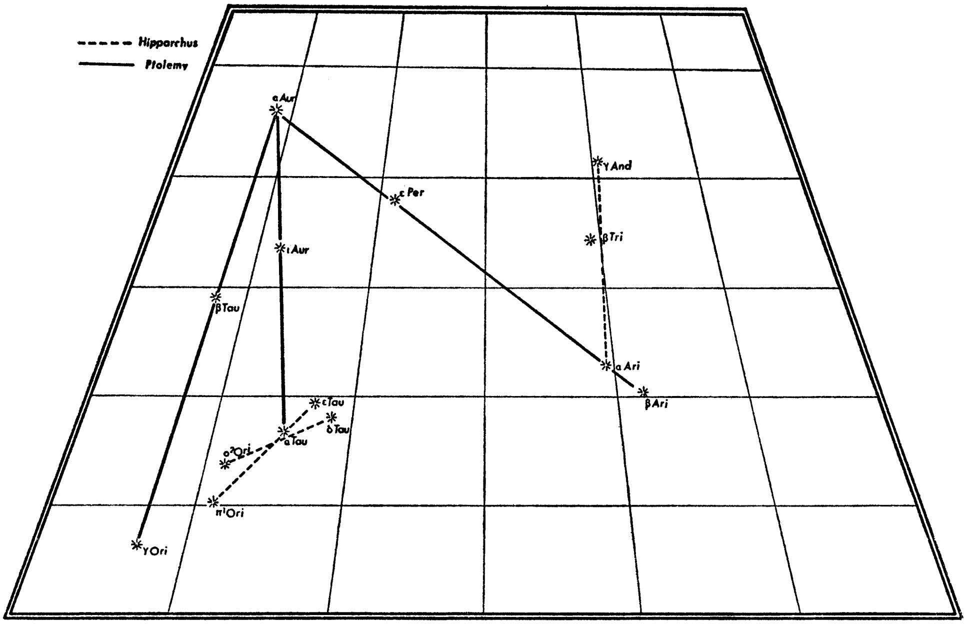


Fig. 8.1

Hipparchus used the method extensively and a number of alignments described by him have survived in the Almagest [VIII, 1; Hei 2, 4]. Most of them are of a very straightforward type, as when it is stated (in modern notation) that *the stars* μ *Leonis,* ε *Leonis, and* ω *Hydrae are seen on a straight line* or that *A line from* α *Canorum venaticorum to* β *Leonis passes through* 7 *and* 15 *Comae Berenicis.*
Others are more sophisticated and introduce an element of measurement based on simple estimates of distances. For instance it is said that α *Canorum venaticorum lies* 1 *digit to the West of a line from* β *Leonis to* η *Ursae maioris.*

Here 1 digit (*daktylos*) is twice the digit used for measuring eclipses (see page 231), viz. $\frac{1}{24}$ of 1 celestial cubit (*pecus*) of 2°, that is 0°;5.

More than twenty alignments due to Hipparchus are quoted by Ptolemy, who declares in very definite terms that he has checked them without finding one single one among them changed during the 260 years separating him from Hipparchus [VII, 1; Hei 2,8]. In order that after a much longer interval of time his successors may have more comprehensive material at hand, he adds a similar number of alignments first observed and described by himself. Since the latter alignments are new they cannot be verified by comparison with earlier ones, but Ptolemy asserts that they agree well with the positions of the stars on the celestial globe of Hipparchus; accordingly a specimen of the latter must have been preserved until Ptolemy's time and available to him, presumably in the Alexandrian Museion.[2])

2) On the stellar globe as a measuring instrument, see Vogt 1920, pp 45-51.

The obvious inference is that all the stars included in the Hipparchian alignments have kept their relative positions unchanged during a period of at least 260 years prior to Ptolemy's time. This result is generalized with respect to both space and time so that Ptolemy defines the fixed stars as those stars which keep their configurations and mutual distances unchanged throughout eternity [VII, 1; Hei 2, 2]. Therefore, to a Ptolemaic astronomer a star with a perceptible proper motion would not be a fixed star, but a planet, or an atmospheric phenomenon.

The Precession of the Fixed Stars

However, Ptolemy's main purpose in studying stellar alignments was not to disprove the idea of proper motion of individual stars. On the contrary, he wanted to show that Hipparchus was wrong in supposing that the stars outside the zodiac remained stationary while the stars inside it shared a common, slow motion towards the East [VII, 1; Hei 2, 3]. This hypothesis of Hipparchus is disproved by the fact that some of the alignments include stars far beyond the zodiac, connecting them with those inside it. Thus Ptolemy's first contribution to the theory of precession is the clearly stated insight that precession is a general phenomenon, affecting all the fixed stars to the same extent.

Ptolemy gives a very brief account of how Hipparchus discovered precession by comparing his own observations of Spica (α Virginis) with those of Timocharis [VII, 2; Hei 2, 12] and Aristyllus [VII, 1; Hei 2, 3]. This showed that *Spica was formerly 8° West of the autumnal equinox but now only 6°*, as Ptolemy quotes verbatim from Hipparchus' (lost) book *On the Changes of the Solstitial and Equinoctial Points*. He also quotes [VII, 2; Hei 2, 15] Hipparchus' result, that these points move against the signs (i.e. westwards) at a rate of at least 1/100° per year, as stated in the book *On the Length of the Year* (page 131). Thus the Hipparchian rate of precession is at least 36″ per year, but on the other hand probably much less than the value 50″.4 accepted to-day. The reservation expressed by the words 'at least' shows that Hipparchus did not take the value too seriously, and that he thought the true value might be greater.

Stellar Observations in the Almagest

The simple method of alignments is insufficient when we are in need of exact positions of the fundamental stars. To determine such positions Ptolemy describes another method, which has since become one of the classical procedures in observational astronomy. It is based on the use of his *astrolabon* (see page 183) and was therefore presumably invented by Ptolemy himself; perhaps he got the idea from his study of occultations of the fixed stars, such events being observed already in 294 B.C. by Timocharis of Alexandria [VII, 3; Hei 2, 28] and also in A.D. 98 by Menelaus in Rome

[VII, 3; Hei 2, 30]. Briefly speaking the procedure is the following: At a certain time t_1 just before sunset the apparent elongation of the Moon from the Sun is observed with the instrument. At a later time t_2 shortly after sunset (when the stars are visible) the difference of longitude between the Moon and a certain fixed star is measured with the same instrument. From these data the longitude of the star with respect to the vernal equinox can be determined. Its latitude is measured directly with the instrument.

The method seems rather complicated but has obvious advantages. First, it results in an absolute determination of the longitude of the star, relating it directly to the vernal equinox through the positions of the Sun and the Moon. Second, it is based only on measurements of differences of longitudes, so that small errors in the mounting and adjustment of the instrument are less important. On the other hand it presupposes that the theories of the motion of the Sun and the Moon (including the theory of parallax) are sufficiently exact to provide reliable theoretical values of the longitudes of the two luminaries at the times t_1 and t_2. This is one of the weak points of the method as used by Ptolemy, considering the rather poor degree of exactness obtainable by his planetary theories. Another weak point is the fact that the Sun is observed near the horizon, where there is a great atmospheric refraction which Ptolemy was unable to take into account (see page 42).

The Observation of Regulus

As an example of the method described above we shall consider two observations of Regulus (α Leonis) which Ptolemy claims to have made in Alexandria on the 9th of Pharmuti in the second year of Antoninus, that is, A.D. 139 Feb 23 [Appendix A No 83, cf. VII, 2; Hei, 2, 14]. Figure 8.2 shows the southern heavens at the time of the observations, with the horizon, the meridian and the equator in fixed positions while the ecliptic is shown in two consecutive positions corresponding to the times of the two observations[3]). The procedure is the following.

First observation. – The time t_1 is given as sunset, but also as $5\frac{1}{2}^h$ in the afternoon; it is not stated whether this is found by some kind of clock, or by the calculated length of the day on the date in question (page 103). – The apparent Sun is at the intersection S_1 between the ecliptic and the horizon. Ptolemy states that it is sighted at Pisces 3°, but since no stars are visible and the instrument is unable to provide absolute longitudes, this value must be derived from the theoretical longitude cal-

3) Copernicus (*De Rev.* II, 14) quotes the Regulus observation by means of which *the way was laid open to the other fixed stars* (fol. 46 r). But he has the time wrong (5 hours after noon instead of $5\frac{1}{2}$ hours) and gives the position of the Sun as $3\frac{1}{24}$° Pisces instead of about 3°. Tycho Brahe gave a general, critical analysis of Ptolemy's method, underlining, among other things, Ptolemy's disregard of atmopheric refraction (see *Progym.* I, 139 ff. = *Opera Omnia* II, 151 ff.). – Delambre (II, p. 248 f.) got some of the fractions wrong, working from a faulty MS. – Manitius (1905, p. 406) underlines the importance of the Regulus observation as a school example of the use of the astrolabon.

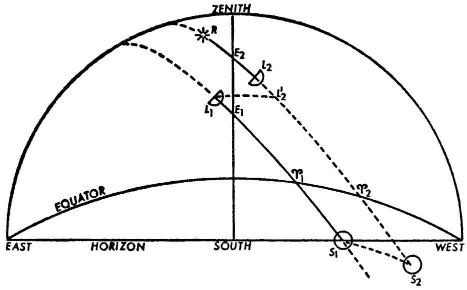


Fig. 8.2

culated below. – Also the statement that the longitude of the *medium coeli* M_1 (the culminating point of the ecliptic, see page 114) is $\lambda(M_1) = 60°$ must be of a theoretical nature and derived from (4.30) by the table of oblique ascensions. – The apparent Moon is at L_1, which is shown as a point on the ecliptic; in fact the Moon has a latitude of 3°;51 South, but this is of no importance to the following. The elongation of the Moon from the Sun is measured with the instrument and found to be

$$\lambda^*_{☽}(t_1) - \lambda^*_{\odot}(t_1) = S_1L_1 = 92\tfrac{1}{8}° \; (= 92°;7,30) \tag{8.1}$$

where the asterisks here and in the following indicate apparent positions. In the following calculations we shall render this result as 92°;7,30 although it goes without saying that Ptolemy's instrument was unable to provide such a degree of accuracy.

In the interval between the two observations two things happen. First, all points on the celestial sphere move from East to West because of the diurnal rotation. In half an hour the displacement is $7\frac{1}{2}°$. It follows everywhere an arc parallel to the equator; thus the Sun moves from S_1 to S_2, the apparent Moon is carried from L_1 to L'_2, and the vernal equinox changes its position from $♈_1$ to $♈_2$. Second, because of its proper motion towards the East the apparent Moon is carried to the position L_2 instead of the point L'_2.

Second observation. – Here the time t_2 is half an hour after sunset, or 6^h after noon. The Sun is invisible at S_2, but the star Regulus has become visible at R; since Regulus has a latitude of 0°;10 North the point R is situated very nearly on the ecliptic. – Now Regulus is observed to have a distance

$$L_2R = \lambda_R(t_2) - \lambda^*_{☽}(t_2) = 57\tfrac{1}{6}° \; (= 57°;10) \tag{8.2}$$

from the apparent Moon, measured eastwards along the ecliptic. – Further, Ptolemy gives the longitude of the new *medium coeli* M_2 as $\lambda(M_2) = 67\frac{1}{2}°$; this is a rough value found by adding the displacement of $7\frac{1}{2}°$ to the longitude of M_1. A more exact value found by (4.30) is $\lambda(M_2) = 67°;10$.

Reduction of the observations

The problem is to determine the longitude $\lambda_R(t_2)$ of Regulus at the time t_2 of the second observation from the given data, that is the times t_1 and t_2 and the two arcs

(8.1) and (8.2). Ptolemy's account of how this was done is very brief and has given rise to some misunderstanding. It seems, however, that his procedure can be explained on the basis of the following formula:

$$\lambda_R(t_2) = \lambda_{\odot}(t_1) \qquad (8.3\text{ a})$$
$$+ [\{\lambda^*_{☽}(t_1) + \pi_1\} - \lambda_{\odot}(t_1)] \qquad (8.3\text{ b})$$
$$+ 0°;15 \qquad (8.3\text{ c})$$
$$+ [\lambda_R(t_2) - \{\lambda^*_{☽}(t_2) + \pi_2\}] \qquad (8.3\text{ d})$$

Here the term (8.3 a) is the true longitude $\lambda_{\odot}(t_1)$ of the Sun at the time t_1 which must be determined by the solar theory. The time interval between the standard epoch t_0 (see page 151) and t_1 is

$$t_1 - t_0 = 885^a 218^d 5\tfrac{1}{2}^h \qquad (8.4)$$

By the usual procedure (page 153) we then calculate the

mean longitude at t_0	330°;45
mean motion in $(t_1 - t_0)$	359°;55,17
mean longitude at t_1	330°;40,17
apogee at t_1	65°;30
mean argument (anomaly)	265°;10,17
equation of centre	+2°;23
true longitude at t_1	333°; 3,17

Leaving out the seconds, Ptolemy gives the result as

$$\lambda_{\odot}(t_1) = 333°;3 \qquad (8.5)$$

This is the theoretical (geocentric) value of the true longitude of the Sun without respect to its parallax. However, Ptolemy continues as if (8.5) were the apparent longitude $\lambda^*_{\odot}(t_1)$ of the Sun. In other words, he disregards the solar parallax the value of which would be a little less than 0°;3 according to the table of parallax [V, 18; Hei 1, 442], the ecliptic being almost perpendicular to the horizon. Considering the accuracy of these observations this is a perfectly reasonable approximation. On the other hand Ptolemy stated that the Sun was s i g h t e d at $\lambda^*_{\odot}(t_1) = 333°$. This value is, perhaps, derived by subtracting the horizontal parallax from (8.5), but is without importance for the following where $\lambda_{\odot}(t_1)$ is used instead of $\lambda^*_{\odot}(t_1)$. It may be noticed that according to Tuckerman's Tables the longitude of the Sun was $\lambda_{\odot}(t_1) = 333.79° = 333°;47$. Accordingly Ptolemy's theoretical value is about $\frac{3}{4}°$ too small. This illustrates the shortcomings of his solar theory, and it is no doubt the principal source of the errors of the longitudes of the fixed stars discussed below (page 253).

The term $\{\lambda^*_{☽}(t_1) + \pi_1\}$ is the true longitude of the Moon at the time t_1, expressed as the apparent longitude plus the parallax in longitude π_1. Consequently (8.3 b) is the true elongation of the Moon from the Sun at the time of the first observation. In a similar way $\{\lambda^*_{☽}(t_2) + \pi_2\}$ is the true longitude of the Moon at the time t_2 and (8.3 d) the angular distance between Regulus and the true Moon.

Finally (8.3 c) is the value adopted by Ptolemy for eastward motion of the Moon during the time interval between the two observations. Without determining the instantaneous velocity of the Moon by the method described above (page 225) Ptolemy satisfies himself by saying that the Moon moves about $\frac{1}{2}°$ per hour and consequently about 0°;15 during the half hour between the two observations.

We can now re-write (8.3) in the form

$$\begin{aligned} \lambda_R(t_2) = {} & \lambda_{\odot}(t_1) && (8.6\text{ a}) \\ & + [\lambda^*_{☽}(t_1) - \lambda_{\odot}(t_1)] && (8.6\text{ b}) \\ & + 0°;15 && (8.6\text{ c}) \\ & - (\pi_2 - \pi_1) && (8.6\text{ d}) \\ & + [\lambda_R(t_2) - \lambda^*_{☽}(t_2)] && (8.6\text{ e}) \end{aligned}$$

which is more adapted to the actual observational procedure. Here (8.6 a) is given by (8.5), and (8.6 e) by (8.2), while (8.6 b) is given by (8.1) with the approximation $\lambda^*_{\odot}(t_1) \approx \lambda_{\odot}(t_1)$. This leaves the term $(\pi_2 - \pi_1)$, or the lunar parallax in longitude at the second observation relative to that at the first, for which Ptolemy adopts the approximate value

$$\pi_2 - \pi_1 \approx \tfrac{1}{12}° = 0°;5 \tag{8.7}$$

without explaining why. Applying these values to (8.6) we find

$$\begin{aligned} \lambda_R(t_2) = {} & 333°;\ 3 \\ & +\ 92°;\ 7,30 \\ & +\ \ 0°;15 \\ & -\ \ 0°;\ 5 \\ & +\ 57°;10 \\ \hline & = 122°;30 \end{aligned} \tag{8.8}$$

where the seconds of arc have been discarded.

This reduction calls for some comments. First, Ptolemy asserts that the position of the Moon at the first observation agrees *with the hypotheses*. This implies that he must have used the lunar theory to check the 'observed' position of the Moon (that is the position derived from the observed elongation and the theoretical longitude of the Sun), or

$$\begin{aligned} \lambda^*_{☽}(t_1) = {} & 333°;\ 3 \\ & +\ 92°;\ 7,30 \\ \hline & \ \ 65°;10,30 \end{aligned} \tag{8.9}$$

where Ptolemy discards the seconds. Using the standard procedure (page 198) and the same time interval as before we find the theoretical value of the geocentric longitude of the Moon from the

mean longitude at t_1	57°;42, 5
mean argument	276°; 3,53
mean elongation	87°; 1,48

centrum		174°; 3,36
equation of centre	+	2°;10
true argument		278°;13,53
equation of argument	+	7°;20

The result is

$$\lambda_{☽}(t_1) = 65°;2,5 \tag{8.10}$$

which is 0°;8,25 less than the apparent, 'observed' value, the Moon being East of its theoretical position. The same difference appears if we calculate the theoretical elongation, adding to the mean elongation the equation of argument of the Moon and subtracting the equation of centre of the Sun. This gives

$$\lambda_{☽}(t_1) - \lambda_{\odot}(t_1) = 91°;58,48 \tag{8.11}$$

which is 0°;8,42 less than (8.1). Considering the uncertainty of the seconds we must conclude that according to the lunar theory the Moon was about 8 or 9 minutes of arc East of its theoretical, geocentric position at the time of the first observation. The problem is whether Ptolemy was prepared to disregard this discrepancy when he said that the observation was in agreement with the theory. Considering the degree of accuracy obtainable from the latter he would have been perfectly justified in doing so; but something points to the conclusion that he thought it possible to account for the 8 or 9 minutes by taking the parallax of the Moon into account. Accordingly we must examine what he said about the parallax of the Moon in relation to the observation of Regulus.

As already mentioned Ptolemy did not explain how he arrived at the relative parallax (8.7). It is possible that it was based on a rough estimate, but also that it was the result of a calculation of the two individual parallaxes π_1 and π_2. The latter hypothesis is strengthened by the fact that Ptolemy quoted the longitudes of the mid-heaven at both of the observations, this parameter being necessary for the computation of the zenith distance of the Moon, and therefore also for the determination of its parallax. To decide this question we have made a very approximate calculation of π_1 and π_2, disregarding the latitude of the Moon (see page 216 f.) and using round values $c = 180°$ and $a_v = 270°$ for the centrum and the true argument respectively. The results can be summarized as follows:

	1st observation	2nd observation
medium coeli E	60°	67°;30
zenith distance of L	10°;15	9°;40
parallax in altitude	0°;16	0°;15
angle between the ecliptic and the vertical through L	77°	70°
parallax in longitude	$\pi_1 = -0°;3,30$	$\pi_2 = -0°;5$

It follows that

$$\pi_2 - \pi_1 = -0°;1,30$$

which compares badly with (8.7). This points to the conclusion that Ptolemy must have relied upon a not very fortunate estimate of the relative parallax[4]). One cannot help feeling that his reduction of the observations is performed in a rather cavalier way and without the painstaking care he so often displays. This was particularly evident in his rough handling of the effect of the parallax of the Moon. But perhaps this apparent carefree attitude has a simple explanation. In order to verify the effect demonstrated by Hipparchus, Ptolemy wanted to compare the longitude of Regulus and other fixed stars at his own time with the corresponding values handed down from Hipparchus almost 300 years earlier. During this interval the total displacement of the fixed stars was found to be almost 3°, so that a small error of a few minutes of arc would be insignificant in view of the accuracy obtainable with his instrumental equipment.

The Rate of Precession

Having thus determined the longitude of Regulus Ptolemy at once compares it with a previous value derived from observations made by Hipparchus in the 50th year of the 3rd Calippic period, that is in 129/128 B.C. (N°46). The two values are

Ptolemy	122°;30
Hipparchus	119°;50
Difference	2°;40

Accordingly it appears that Regulus has increased its longitude with an amount of 2°;40 in 265 years. This is roughly equivalent to 1° per century, which is the rate of precession adopted by Ptolemy after its having been confirmed by a certain number of other observations. He maintains, for instance, that the method described above was applied to Spica (α Virginis) and then to other bright stars near the ecliptic. In some cases the method of observation is so changed that the role of the Moon is taken over by a fixed star, the longitude of which has been already determined. But in all cases the result is stated to be the same: The relative distances in longitude of the stars remain unchanged while their absolute, individual longitudes have increased by about 2°;40 since the time of Hipparchus [VII, 2; Hei 2, 16]. The whole procedure reveals that Ptolemy must have had a fairly large number of stellar positions found by Hipparchus at his disposal. In fact, there is no doubt that Hipparchus compiled a stellar catalogue, now lost, but used by Ptolemy in these investigations and serving as a model for his own catalogue described below (page 249).

4) Manitius maintained that the parallax in longitude was positive East of the meridian (first observation), but negative West of it (second observation); see his commentary to the German translation of the Almagest (vol. 2, p. 397–399). This would give $\pi_2 - \pi_1 \approx -0°;8,30$ and thus explain the difference; but in the given situation both π_1 and π_2 must be negative (westwards). Furthermore, he read the zenith distance off a celestial globe and found it to be circa 15°. From this value he deduced $\pi_1 = +10'$ and $\pi_2 = -5'$, without determining the angles between the ecliptic and the two verticals through L_1 and L_2.

The Axis of the Precessional Motion

So far Ptolemy was able to interpret his observation as showing a regular increase of stellar longitudes with time. But other problems had to be solved before the phenomenon of precession was fully described. For instance, there are some indications that Hipparchus believed the precessional motion away from the vernal equinox to be a property only of the fixed stars situated near the ecliptic, or perhaps of all the stars inside the zodiacal belt stretching about 6° to each side of it (page 239). However, Ptolemy is aware that precession is a general phenomenon in which each and all of the fixed stars take part. This is to him so obvious that he does not waste many words on proving it. Nevertheless, we can discern two different reasons for this belief.

The first is purely phenomenological. The many alignments described above (page 238) show that the fixed stars conserve their mutual distances all over the heavens. Of course this would not be the case if some of them had a particular motion towards the East different from that of the others. The second reason is in a way more interesting since it stems directly from Ptolemy's general cosmological ideas, which in general are not very prominent in the Almagest. In fact Ptolemy refers explicitly to the theory of spheres, assuming all the fixed stars to be part of, or attached to, a solid sphere [VII, 3; Hei 2, 16], with the obvious consequence that they must all of them move together with this sphere as a whole.

The only remaining problem is, therefore, to determine the motion of the stellar sphere in a more precise way. Conceiving this motion as a rotation, the question is to find the axis about which it is performed. Already Hipparchus had made the assumption that the fixed stars near the ecliptic move along the ecliptic, i.e. about the axis of the latter circle. Ptolemy's main preoccupation is to show that his predecessor was right on this point. To obtain greater accuracy in this final part of the investigation he extends the span of years to about 375[a] by taking the previously mentioned occultations observed by Timocharis and Aristyllus of Alexandria into account. The result is a list of the following 18 fixed stars for which declinations are given as determined by Timocharis, Hipparchus, and Ptolemy himself [VII, 3; Hei 2, 20].

$-90° < \lambda < 90°$	$90° < \lambda < 270°$
1. α Aquilae (Altair)	10. α Leonis (Regulus)
*2. η Tauri (Alcyone)	*11. α Virginis (Spica)
3. α Tauri (Aldebaran)	*12. η Urs. Mai. (Benetnasch)
*4. α Aurigae (Capella)	13. ζ Urs. Mai. (Mizar)
5. γ Orionis (Bellatrix)	14. ε Urs. Mai. (Alioth)
*6. α Orionis (Betelgeuse)	*15. α Bootis (Arcturus)
7. α Can. Mai. (Sirius)	16. α Librae (Zuben el Genubi)
8. α Geminorum (Castor)	17. β Librae (Zuben el Chamali)
9. β Geminorum (Pollux)	18. α Scorpii (Antares)

To this quite impressive collection of material Ptolemy now puts three different questions.

First, he asks whether the precessional motion of the stellar sphere affects other coordinates than the longitude of the stars, or more precisely, if there is any influence on their declinations. This proves to be the case, for all the declinations show an unmistakable trend: If a fixed star is situated in the same half part of the heavens as the vernal equinox – or, more precisely, if its longitude lies in the interval $-90° < \lambda < 90°$, its declination increases with time, while it decreases if the star is situated in the remaining part of the heavens [VII, 3; Hei 2, 18]. This result is sufficient to prove that the starry sphere does not rotate about the axis of the equator, for if this were so all the declinations would be constant and independent of time.

Second, Ptolemy investigates the possible influence of the precession on the latitude of the stars. For this purpose he supplements the material by the Roman observations of Menelaus from A.D. 98 (Appendix A, N^{os} 52–53) and also an occultation of the Pleiades observed in A.D. 92 (Appendix A, N^{o} 51) by Agrippa of Bithynia; these occultations are reduced by Ptolemy by means of the lunar theory in the same way as the observation of Regulus. To give a single example, Ptolemy shows from two of the early occultations that both in 294 B.C. and in 283 B.C. Spica had a southern latitude of 2°;0. The corresponding observation by Menelaus in A.D. 98 gave precisely the same result, so that the latitude of Spica had remained constant for almost 400 years. A similar result is obtained for the other stars under consideration and without hesitation Ptolemy generalizes it to all the fixed stars [VII, 4; Hei 2, 34]. It follows that since the precession affects the longitude but not the latitude, it can be interpreted as a motion of the sphere of the fixed stars about the axis of the ecliptic.

Finally, Ptolemy applies the observed declinations to make an independent redetermination of the rate of precession. But for this purpose he selects only the six stars marked by an asterisk in the list above. This provides us with the following data:

Star	Longitude according to Ptolemy	Declination according to		
		Aristyllus	Hipparchus	Ptolemy
η Tauri	33°;40	+14°;30	+15°;10	+16°;15
γ Orionis	54°	+ 1°;12	+ 1°;48	+ 2°;30
α Aurigae	55°	+40°; 0	+40°;24	+41°;10
η Urs. Mai.	149°;50	+61°;30	+60°;45	+59°;40
α Virginis	176°;40	+ 1°;24	+ 0°;36	− 0°;30
α Bootis	177°	+31°;30	+31°; 0	+29°;50

In each case it is possible to determine the change in longitude corresponding to the given change in declination (by means of the table of the obliquity of the ecliptic).

Ptolemy maintains that between Hipparchus' and his own time all these six stars have moved 2°;40 towards the East[5]).

The General Theory of Precession

The final result of the investigations into the nature of the precession of the fixed stars can be summarized in the following two expressions:

$$\lambda(t) = \lambda(t_0) + p(t - t_0) \tag{8.12}$$

$$\beta(t) = \beta(t_0) \tag{8.13}$$

Here p is the rate of precession, supposed to have the constant value of

$$p = 1° \text{ per century} \tag{8.14}$$

if the times t_0 and t in (8.12) are measured in centuries. This calls for at least three comments.

First, the fact that Ptolemy assumed the rate of precession to be constant means that the relations (8.12–13) are correct also in modern astronomy, if only the value of p is changed to the modern value of 50″.23 per year, or about 1°;24 per century, equivalent to 1° in about 71 years.

Second, the fact is that the constant rate of precession adopted by Ptolemy was too small. In the course of time this gave rise to increasing difficulties as observations became more precise and the span of years more extended. In the Middle Ages this led to the belief that the rate of precession had changed since the time of Ptolemy. This belief, together with actually observed changes of the obliquity of the ecliptic was the basis of the Mediaeval theories of the trepidation of the equinoxes[6]).

Third, Ptolemy's cosmological theory of the precession as a motion of a separate starry sphere relative to the vernal equinox reveals his belief in their absolute motion. Today we interpret precession as a westwards motion of the equinoxes relative to a frame of reference defined by the Sun and the fixed stars, and caused by the gyro-

5) Pannekoek has shown (1955, p. 63) that if Ptolemy had used the exact formula

$$\Delta\delta = \text{number of years} \cdot p \cdot \text{sine } \varepsilon \cdot \cos \alpha$$

for the change of declination of a star with the right ascension α he would have found the rate of precession $p = 38''$ per year using only the six stars listed above. The remaining twelve stars would have led to $p = 52''$ per year. Thus Ptolemy may have selected the six stars because they confirmed the rate of precession (i.e. $p \approx 36''$ per year) already found from the change of longitude; nevertheless, he did not suppress the other evidence.

6) There were several forms of the theory of trepidation. They all assumed that the total displacement of the fixed stars over a number of years contained a periodic function of time. The theory was advanced by Thābit ben Qurra in his treatise *On the Motion of the Eighth Sphere* (ed. Carmody, 1960, pp. 84–113). – Cf. Goldstein (1965) and Hartner (1971).

scopic properties of the rotating Earth. To Ptolemy the fixed stars constitute no immovable frame of reference. They really move relative to the equinoctial points, which are regarded as having fixed positions in space. Eventually this led Mediaeval astronomers to the conception of a ninth sphere outside the sphere of the stars, containing no celestial bodies, and partaking only of the daily rotation of all the heavens[7]). Upon this sphere the ecliptic and its poles were supposed to occupy fixed positions (apart from the diurnal movement), so that precession could be interpreted as a slow secular rotation of the 8th sphere relative to the 9th.

The catalogue of the fixed stars

The observation of Regulus described above (page 240) and its subsequent reduction may be regarded as a typical example of many similar investigations which had to be performed before Ptolemy was able to compile his famous catalogue of the fixed stars. The development of the theory of precession was another preliminary step towards the same goal. This catalogue is one of the most remarkable achievements of ancient astronomy as a whole, and has given rise to more research than most other parts of the Almagest. It has been much used by later astronomers and was finally published separately in a critical edition by Peters and Knobel. In the Almagest it is preceded by a general introduction [VII, 4; Hei 2, 34] and takes up two long chapters in two separate books, one containing the stars of the northern heavens [VII, 5; Hei 2, 38] and another the southern stars [VIII, 1; Hei 2, 106] down to low celestial latitudes. In fact, the most southern star listed in the catalogue is Canopus (α Argus), which has a latitude $\beta \approx -75°$ corresponding to a declination of $\delta \approx -52°$. This implies that Ptolemy observed at least one fixed star culminating only about 7° above the horizon of Alexandria.

In the catalogue most of the stars are grouped in constellations while a few are regarded as appended to a constellation without properly belonging to it. The following table shows the Ptolemaic constellations with the number of stars pertaining or appended to them.

7) The idea of a 9th sphere containing the sphere of the fixed stars is already introduced by Ptolemy in the *Hypotheses* (ed. Nix, pp. 122 and 125). It serves as a 'mover' of the starry spheres, but seems to have no direct connection with the theory of precession.

Northern Constellations		*Zodiacal Constellations*		*Southern Constellations*	
Little Bear	7 + 1	Ram	13 + 5	Sea Monster	22
Great Bear	27 + 8	Bull	32 + 11	Orion	38
Dragon	31	Twins	18 + 7	River	34
Cepheus	11 + 2	Crab	9 + 4	Hare	12
Ploughman	22 + 1	Lion	27 + 5	Dog	18 + 11
Northern Crown	8	Virgin	26 + 6	Little Dog	2
Man Kneeling	28 + 1			Argus	45
Lyre	10			Water Snake	19 + 2
Bird	17 + 2	Balance	8 + 9	Bowl	7
Cassiopeia	13	Scorpion	21 + 3	Raven	7
Perseus	26 + 3	Archer	31	Centaur	37
Charioteer	14	Goat	28	Wild Beast	19
Serpentarius	24 + 5	Water Bearer	42 + 3	Censer	7
Serpent	18	Fishes	34 + 4	Southern Crown	13
Arrow	5			Southern Fish	11 + 6
Eagle	9 + 6				
Dolphin	10				
Forepart of Horse	4				
Horse	20				
Andromeda	23				
Triangle	4				

This gives

360 stars North of the zodiac
346 stars inside the zodiac
316 stars South of the zodiac
1022

or a total of 1022 fixed stars. This is a very impressive number, although it is not as exhaustive as Ptolemy believed. In fact he maintained that he had observed, as far as possible, all fixed stars down to the sixth magnitude [VII, 4; Hei 2, 35]. But this is far from being true, as we shall see below (page 259).

The catalogue is laid out in the form of a table with 1022 lines and 4 columns. Column I serves to identify the star, and Column II, III, and IV give its longitude, latitude, and apparent magnitude respectively. Inside the various constellations the stars are not listed according to any strict rule. In Ursa Minor they are arranged according to increasing longitudes, but this principle is broken already in the following (Ursa Major) and in most other constellations. We shall now consider a few of the more general problems inherent in Ptolemy's catalogue, without entering upon all the many questions of detail to which it gives rise.

The Identity of the Stars

Already Column I raises the problem of how the stars mentioned in this column can be identified. Since they are listed according to constellations this implies that we ought

to consider the Ptolemaic constellations in their relationship to both earlier and later configurations of the stars. But this would carry us too far. We remark only that the very notion of a constellation has another meaning for Ptolemy than in modern astronomy. Nowadays it means a certain part of the heavenly sphere inside well defined boundary lines, but without any essential reference to imaginary pictorial representations of a number of stars. Thus Canis Maior is simply an area delimited by a small circle parallel to the celestial equator through points with the declination $\delta = -11°$, another small circle with $\delta = -33°$, and two great circles through the poles of the equator with the right ascensions $\alpha = 6^h10^m$ and $\alpha = 7^h25^m$ respectively. In this way the whole sphere has been completely covered with constellations.

In Ptolemy's time this was not the case. It is obvious from the Almagest that a constellation is primarily an imaginary image projected upon the heavenly sphere and grouping together a certain number of stars into a picture, while other stars are left to themselves without belonging to any constellations. For instance it is a curious fact that Arcturus is not regarded as a part of any constellation, but only loosely appended to the Ploughman, or Bootes, while it is now placed as α Bootis, the modern constellation Bootes being so defined that it includes the stars of the Ptolemaic Ploughman together with Arcturus. This is a rather untypical example, Arcturus being a bright and conspicuous star, while most of the other 'appended' stars in Ptolemy are rather faint. It follows from what was said above that they have all been integrated into existing or newly defined constellations. Thus the 6 stars appended to the Virgin have been absorbed into the constellation Virgo while the 8 stars appended to the Great Bear have been given to three new constellations, viz. 2 to Canes Ven., 3 to Lynx, and 3 to Leo minor. All this means that the fact that a star is placed inside or near one of Ptolemy's constellations does not imply that it is found also in the modern constellation bearing the same name.

Ptolemy has no systematic method of denoting the individual stars inside his constellations, either by letters or by numbers. In fact, he always sees them against the imaginary figure which is the real constellation to him. This forces him to describe the position of a star in words. For instance, α Can. ven. (Cor Caroli) is defined as *The star under the tail* [of the Great Bear] *far to the South*, while *The bright star in the tail* [of the Bird] is α Cygni. In Cassiopeia *The star above the chair along the thighs* is a more dubious example, depending too much on an ill-defined image, but usually identified with α Cass. Such vague descriptions have made the identification of some of the stars in the catalogue rather uncertain and caused disagreement among scholars. A list of differences of identification has been given by Peters and Knobel (1915, p. 114 ff.), whose own results are now generally accepted as final.

It is worth noticing that the tradition of assigning proper names to individual stars is not very far advanced in Ptolemy. It belongs mainly to the Arabs, and in the Almagest the verbal descriptions of stellar positions inside the constellations are supplemented by the use of a proper name only in a few cases, such as Arcturus, Capella, Regulus, Spica, Antares, the Dog (Sirius), Procyon, Canopus, and perhaps a few others.

The Accuracy of the Catalogue

The only way of checking the accuracy of the longitudes and latitudes listed in Columns II and III respectively is to compare them with correct stellar positions determined for the epoch of the catalogue. Ptolemy states explicitly that the epoch is the beginning of the reign of Antoninus [VII, 4; Hei 2, 36], or A.D. 138. Accordingly we ought to determine the positions of the stars in the catalogue for that date. The most careful investigation of this kind has been made by Peters and Knobel, who proceed in a slightly different way. Using Piazzi's catalogue (epoch A.D. 1800) and taking both precession and proper motions into account, they calculated the positions of all the 1022 stars for the year A.D. 100 in order to verify the epoch given by Ptolemy.

We cannot here go into all the details of this very comprehensive study, but shall restrict ourselves to a small survey of the principal reference stars used by Ptolemy and mentioned above (page 236). They are listed in the following table. Here Column I gives the name of the star; Column II is its longitude as given in the Almagest according to Peters and Knobel, whose critical edition is based on a collation of 26 MSS. Column III is the longitude as calculated for A.D. 100, and Column IV is the difference between the calculated and the Ptolemaic longitudes. Columns V, VI and VII are arranged in the same way but refer to latitudes. Finally Column VIII contains the order of magnitude given in the Almagest while Column IX contains magnitudes taken from the Harvard Revised Photometry.

I	II λ(Ptol)	III λ(calc)	IV Δλ	V β(Ptol)	VI β(calc)	VII Δβ	VIII m(Ptol)	IX m(mod)
α Tauri	42°;40	43°;20	+40′	− 5°;10	− 5°;37	−27′	1	1.1
β Tauri	55°;40	56°; 9	+29′	+ 5°; 0	+ 5°;14	+14′	3	1.8
β Aurigae	62°;50	63°;31	+41′	+20°; 0	+21°;15	+75′	2	2.1
α Geminorum	83°;20	83°;52	+32′	+ 9°;40	+ 9°;55	+15′	2	2.0
β Geminorum	86°;40	87°; 5	+25′	+ 6°;15	+ 6°;31	+16′	2	1.2
α Leonis	122°;30	123°;31	+61′	+ 0°;10	+ 0°;24	+14′	1	1.3
γ Virginis	163°;10	163°;59	+49′	+ 2°;50	+ 2°;58	+ 8′	3	3.6
α Virginis	176°;40	177°;26	+46′	− 2°; 0	− 1°;46	+ 4′	1	1.2
α Librae	198°; 0	198°;41	+41′	+ 0°:40	+ 0°;35	− 5′	2	2.8
β Scorpii	216°;20	216°;46	+26′	+ 1°;20	+ 1°;15	− 5′	3	2.9
α Scorpii	222°;40	223°;20	+40′	− 4°; 0	− 4°;20	−20′	2	1.2
α Capricorni	277°;20	277°;23	+ 3′	+ 7°;20	+ 7°;10	−10′	3	3.4
β Capricorni	277°;20	277°;37	+17′	+ 5°; 0	+ 4°;49	−11′	3	3.2
δ Capricorni	296°;20	297°; 1	+41′	− 2°; 0	− 2°;15	−15′	3	3.0

In this table the positions are given in degrees and minutes. This is according to modern usage, but misleading as to the accuracy of the underlying observations. In fact, in the catalogue, Ptolemy gives the longitudes and latitudes in degrees and simple fractions of one degree. These fractions are only seven in number, viz.

$\frac{1}{2}°$ $\frac{1}{4}°$ (a)

$\frac{2}{3}°$ $\frac{1}{3}°$ $\frac{1}{6}°$ (b)

and the two combinations

$$\tfrac{3}{4}° = \tfrac{1}{2}° + \tfrac{1}{4}° \quad \text{and}$$

$$\tfrac{5}{6}° = \tfrac{1}{2}° + \tfrac{1}{3}°$$

This is the reason why the following number of minutes

$$10', 15', 20', 30', 40', 45', 50'$$

appears so frequently in the modern versions of the catalogue. It shows that it would be illusory to ascribe to Ptolemy an observational accuracy better than 10′ or 15′.

There have been some speculations as to the origin of these fractions. The fractions (a) seem to be found with an instrument with divisions of $\frac{1}{4}°$, and the fractions (b) with another instrument with divisions of $\frac{1}{6}°$. A single instrument producing both sets without judging would then have had to be divided into $\frac{1}{12}°$. But no such conclusions can be drawn. We do not know how Ptolemy's instrument was graduated; but even if it was divided into whole degrees only, the fractions above would be those which would most naturally come into the mind of an observer reading off a position. Here one cannot help noticing that the sets (a) and (b) played a particular role in ancient Egyptian mathematics, and that they still had a certain importance in everyday Greek reckoning.

The Errors in Longitude

The most notable characteristic of the table is that all the differences in Col. IV are positive. This implies that the longitudes of the 14 reference stars of our table are without exception smaller than the longitudes calculated for the year A.D. 100. This general feature appears also in the rest of the stars of the catalogue, of which only 142, or 14%, show greater longitudes. It cannot be explained by the fact that the epochs of Ptolemy's catalogue (A.D. 138) and Peters and Knobel's list (A.D. 100) are different, for in 38 years the true precession would increase the longitudes in Column III by

$$38 \cdot 49''.86 = 32'$$

and consequently increase the differences in Column IV by the same amount. The mean value of these differences is 35′. Adding this to the 32′ of precession we get 1°;7. In other words the 14 stars of our table are on the average 1°;7 behind their true positions. Considering all the stars of the catalogue the same result is confirmed: Ptolemy's longitudes are about 1° too small. Accordingly there must be a systematic error in the catalogue as far as longitudes are concerned; but it is by no means easy to determine its cause although a considerable amount of attention has been given to this particular problem.

One possible solution rests on the assumption that for some reason or other Ptolemy gave a wrong epoch to his catalogue, and that the true epoch should be some years before A.D. 100 (cf. Tannery 1893, p. 270). The mean value 35′ of the difference in Column IV corresponds to an effect of the true precession in

$$\frac{35 \cdot 60}{49.86} = 42.11 \text{ years}$$

Thus the mean deviation would disappear if we assume that the catalogue was calculated for an epoch 42 years before A.D. 100, that is for A.D. 58. Exactly the same result follows if all the stars of the catalogue are taken into account and not only the 14 fundamental stars of the table here above.

This explanation is indeed just as simple and tempting as it is without foundation in the text and contradictory to Ptolemy's explicit statement that he used the first year of Antoninus as epoch of the catalogue [VII, 4; Hei 2, 36]. Therefore Peters and Knobel tried to support this hypothesis by another argument. We saw above (page 245) that Ptolemy compared his Regulus observation with one made by Hipparchus in 129 B.C. From 129 B.C. to A.D. 58 is a span of 186 years, in which the true precession of the stars would increase their longitudes by $49''.86 \cdot 187 = 2°;35$. But this latter number is very nearly equal to the 2°;40 adopted by Ptolemy as the total amount of precession from Hipparchus' time to his own (page 245). The two authors therefore conclude that *the catalogue is simply that of Hipparchus modified by a constant added to the longitude* (Peters and Knobel, p. 15).

This hypothetical explanation of the error in longitude is not new in the history of astronomy. Already certain Muslim astronomers inclined to the view that Ptolemy simply took over the results of his predecessors, correcting them by an amount derived from his false constant of precession[8]). In later times both Delambre (I, p. 183), Tannery (1893, p. 270), Bjørnbo (1901, p. 210) and Boll (1901) have adopted more or less the same position, and with Peters and Knobel's edition it has achieved an almost canonical status. On the other hand Laplace (1796, p. 235), Dreyer[9]) and Vogt (1925) have defended Ptolemy against the accusation of having been dishonest when he compiled the catalogue, suppressing or distorting the results of earlier astronomers.

Admittedly there are many obscurities in Book VII of the Almagest, and it will be an extremely difficult task to give a detailed solution of the problem of how Ptolemy produced his catalogue. Given the state of the source material it is even an open question if any satisfactory solution is possible. An exhaustive discussion of all the material would certainly exceed the limits of the present book, and we shall restrict

8) Bjørnbo (1901) infers from statements by al-Ṣūfī (903–986) and al-Battānī (died 929) that Ptolemy relied upon a catalogue of the fixed stars made by Menelaos. There is not sufficient evidence that Menelaus ever compiled such a catalogue, although he may have determined a considerable number of stellar longitudes.

9) Dreyer changed his opinion on this point. In his *History* (1906, p. 202) he considered Ptolemy's catalogue as *nothing but the catalogue of Hipparchus brought down to his own time with an erroneous value of the constant of precession.* But in his two papers in the Monthly Notices of the Roy. Astr. Soc. (1917 and 1918) he arrived at the result that there is *no reason to disbelieve his positive statement, that he had made extensive observations of fixed stars* (1917, p. 539).

ourselves to pointing out that the hypothesis of Ptolemy as a mere plagiarist of Hipparchus rests upon at least two primary suppositions, viz.

1) that Hipparchus had compiled a catalogue with longitudes and latitudes of at least all the stars listed by Ptolemy, and

2) that Ptolemy deliberately misled his readers by his very emphatic statement that his catalogue was the result of his own observations with the astrolabon [VII, 4; Hei 2, 35].

Ptolemy and Hipparchus

With regard to the first point we have already seen (page 245) that Ptolemy had access to a considerable number of stellar positions found in Hipparchus or derived from the observations of the latter (page 247). He also speaks of Hipparchus' *records of the fixed stars* [...] *which have come down to us in an excellent condition* [VII, 1; Hei 2, 3]. Although this statement is made in connection with the alignments described at the beginning of this chapter (page 237) there is no doubt whatever that Ptolemy was able to draw upon material from Hipparchus' own hand. The only question is if this material had such a character that Ptolemy could transform it into his catalogue by adding a constant to the Hipparchian longitudes? What were the contents of this famous and often praised, but unknown stellar catalogue of Hipparchus?

Let us first consider the only completely extant work by Hipparchus, his commentary on the *Phainomena* of Aratus and Eudoxus, which has been edited by Manitius and very carefully examined by H. Vogt (1925). It contains among other things a description of 42 constellations; for 374 stars there are a total of 859 numerical data; 22 more are conserved in Ptolemy and Strabo. For 252 of these stars the given data are equivalent to only one coordinate. This leaves 122 stars for which 2 coordinates can be found without extra hypotheses. The data are of various kinds. In 64 cases Hipparchus gives declinations and in 67 cases right ascensions. In 340 cases he states the point of the ecliptic culminating together with the star, while the rest of the data are related to risings and settings.

It is a remarkable fact that all this material contains no longitudes. In fact, we know Hipparchian longitudes only for Regulus and Spica, and that only because they are quoted by Ptolemy. Obviously Hipparchus preferred either equatorial or mixed coordinates at the time when he wrote the commentary, but not longitudes and latitudes. This proves that Ptolemy could not have used this work directly when he compiled his catalogue. The question is whether he used it indirectly, that is, whether he calculated longitudes from the Hipparchian data and added 2°;40 to the resulting values.

This problem has been attacked and solved by Vogt, who proceeded exactly as Ptolemy might have done, calculating 'Hipparchian' longitudes and latitudes for the 122 stars for which these two coordinates can be found independently of supple-

mentary assumptions. As an illustration we shall consider his results for those among the fundamental reference stars listed above (page 236) which are included among the 122 stars under consideration. This excludes β Aurigae, β Geminorum, γ Virginis, and α and β Capricorni. For the remaining stars the results are listed in the following table in which

Column I gives the name of the star
Column II lists the true length λ_1 at the time of Hipparchus, or 130 B.C.
Column III is the length l_1 calculated from the Hipparchian material
Column IV gives $\Delta\lambda_1 = \lambda_1 - l_1$
Column V gives the true length λ_2 at the time of Ptolemy, or A.D. 137, and
Column VI is the length l_2 given by Ptolemy in the Almagest

All the values are given in degrees to two decimal places.

I	II λ_1	III l_1	IV $\Delta\lambda_1$	V λ_2	VI l_2	VII $\Delta\lambda_2$	VIII Δl
α Tauri	40.20	39.76	+0.44	43.84	42.67	+1.17	2.91
β Tauri	53.01	52.90	+0.11	56.66	55.67	+0.99	2.77
α Geminorum	80.73	81.21	−0.48	84.38	83.33	+1.05	2.12
α Leonis	120.38	119.64	+0.74	124.03	122.50	+1.53	2.86
α Virginis	174.30	174.04	+0.26	177.94	176.67	+1.27	2.63
α Librae	195.54	195.23	+0.31	199.19	198.00	+1.19	2.77
β Scorpii	213.63	213.00	+0.63	217.28	216.33	+0.95	3.33
α Scorpii	220.20	219.92	+0.28	223.84	222.67	+1.17	2.75
δ Capricorni	293.88	293.73	+0.15	297.53	296.33	+1.20	2.60

This material comprises only 9 stars and is of course quite insufficient. Nevertheless, it can serve as a pointer to the general conclusions reached by Vogt by means of all the 122 stars. Thus Column IV shows that the longitudes derived from Hipparchus' observations are, in general, greater than the true ones, just as Column VII shows the same to be the case in Ptolemy, as we know already (page 253). But the fact that $\Delta\lambda_1$ is considerably smaller than $\Delta\lambda_2$ proves that Hipparchus was a better observer than Ptolemy; it is impossible to decide if this was because he was more careful or had better instruments[10]).

The mean value of $\Delta\lambda_2$ in Col. VII is $1°.17 = 1°;10$. This agrees fairly well with the $35' + 32' = 1°;7$ which according to page 253 represents the mean error of the longitudes of all the reference stars tabulated on page 236 (cf. Figure 8.3).

Finally Column VIII reveals the differences between the Ptolemaic and the 'Hipparchian' longitudes. The mean value is $2°.75 = 2°;45$ or almost the same as the

10) The hypothesis that the errors of Ptolemy's observations might be due to an eccentricity of the ecliptic ring of the astrolabon has been examined in great detail by A. Czwalina (1958, p. 287 ff.). Also Dreyer (1917) called attention to the possible errors of the instrument, and to the disastrous consequences of ignoring refraction when observing the Sun at the horizon.

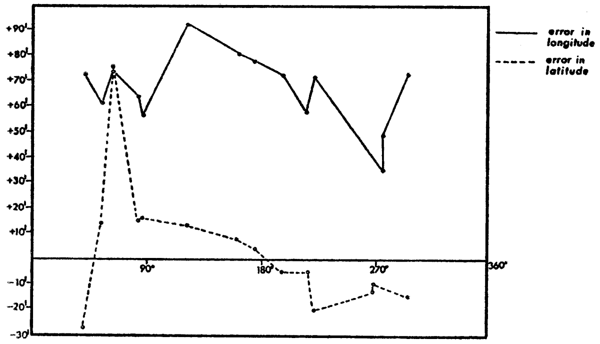


Fig. 8.3

2°;40 which Ptolemy stated to be the increase in longitude from Hipparchus to his own time. Taking all the 122 stars into account Vogt found the mean increase Δl to be $2°.52 = 2°;31$. But the important thing is that the values of Δl are not constant as they ought to be if Ptolemy had added a constant to the Hipparchian longitudes. This is seen already from the few values of Column VIII, but appears much more strikingly if all the 122 stars are considered. In fact, Δl varies between $+5°.56$ in μ Cephei and $-3°.94$ in μ Draconis.

The Rehabilitation of Ptolemy

This brief summary of Vogt's investigations is sufficient to justify his general conclusion that the discrepancies between the Ptolemaic and the 'Hipparchian' longitudes are too great to be explained on the assumption that Ptolemy simply copied Hipparchus' catalogue with an additional constant.

The same conclusion can be supported by a number of other reasons. Most convincing is the investigation of the latitudes in Ptolemy's catalogue made by Vogt, who compared them with latitudes derived from Hipparchus in much the same way as the longitudes. Also here the discrepancies were, in general, too great to be consistent with the hypothesis of Ptolemy as a mere plagiarist. We shall not go into the details of this question, but only mention the fact that the Ptolemaic latitudes listed in the table on page 252 show a systematic error of a different kind from that of the longitudes. As seen from Column VII, the difference $\Delta\beta = \beta(\text{calc.}) - \beta(\text{Ptol.})$ is positive for stars with longitudes from (roughly) 0° to 180°, but negative on the other half of the ecliptic. As suspected already by Delambre, this is most easily explained on the assumption that Ptolemy used an erroneous value for the geographical latitude of Alexandria in conjunction with his not quite correct value of the obliquity of the ecliptic.

Finally, it is possible to draw an important inference merely from the number of stars. Ptolemy's catalogue lists 1022 fixed stars. If the plagiarist hypothesis were true, then the Hipparchian material must have comprised an equal, or greater number. In

1901 F. Boll showed that Hipparchus could not have catalogued more than about 850 stars. This left about 170 stellar positions which must necessarily be due to later astronomers, and probably to Ptolemy himself. As we have seen, H. Vogt restricted the number of Hipparchian stars with determinable coordinates even more, thus leaving an even greater scope for Ptolemy's own contributions.

This vindication of Ptolemy as an independent astronomer rests upon one fundamental presupposition, namely that we can evaluate the achievement of Hipparchus on the basis of his extant commentary to Aratus. But can one be sure that he did not make – at a later time – a new survey of the heavens, resulting in a new catalogue in which the stars were characterized by longitudes and latitudes? and that this catalogue was not used by Ptolemy in the way indicated by Peters and Knobel, and the previous supporters of their hypothesis?

There is, indeed, at least one indication that Hipparchus later made another survey of the heavens. In his *Natural History* (II, 24, 95) Pliny relates that Hipparchus detected a new star, and that this caused him to undertake a survey of the heavens in order to see if such changes occurred more frequently. Further, Pliny maintains that for this purpose Hipparchus devised certain instruments by which the location and magnitudes of the fixed stars could be ascertained. Thus 'he left the heavens as a bequest to all' (*caelo in hereditato cunctis relicto*). If this new star is the same as the nova mentioned in Chinese records from 134 B.C. then this second survey of the heavens belongs to the later years of Hipparchus' scientific activity (cf. Fotheringham 1919).

Now Pliny is a late and not too reliable source, and there is no reason to believe that he had any intimate knowledge of Hipparchus' archievements. If we take his rather vague account as a testimony of a second survey of the heavens made by Hipparchus and resulting in a catalogue of longitudes and latitudes, it is at the cost of destroying the image of Ptolemy as a scholar who conscientiously quoted his sources. Is it conceivable that he deliberately suppressed the existence of such a catalogue, and concealed his use of it? As mentioned before (page 13) Ptolemy is, in general, more careful in paying respect to his predecessors than any other scientific writer in Antiquity. Especially is *the Divine Hipparchus* almost always held in great esteem. Accordingly, there is little reason to suppose that Ptolemy departed from his usual policy at this particular point. Our general impression of his moral and intellectual integrity would be damaged beyond repair if we had to believe that he simply derived his catalogue from a previous work by Hipparchus without the slightest acknowledgement of the fact. It is a much more reasonable assumption that the survey mentioned by Pliny is that of the commentary.

Not all the arguments set forth above are possessed of the same weight. Taken together they seem, however, to offer convincing evidence for the conclusion that Ptolemy was entitled to present his catalogue of the fixed stars as a result of his own observational work. In return we must acknowledge that he was perhaps not as good an observer as Hipparchus, and that his erroneous equinox may account for many of the errors. But the fact that the one was possibly a more diligent observer does not make the other a scientific fraud.

The Magnitudes of the Stars

A remarkable feature of Ptolemy's catalogue is the order of magnitude attached to the individual stars as an attempt to characterize their apparent brightness by a number. It is now established that the classification of all the stars visible to the naked eye in 6 orders of magnitude goes back at least to Hipparchus. But it is obvious that before the introduction of exact photometric methods this classification had to be rather arbitrary. Nevertheless, the table on page 252 shows that the magnitudes given in the Almagest correspond fairly well to modern values as far as the brighter stars are concerned, the most notable exception to this rule being β Tauri[11]). We shall here compare the number of stars as placed in the different orders of magnitude by Ptolemy with a modern count comprising all stars visible to the naked eye with a declination $\delta > -35°$, that is, situated in roughly the same part of the heavens as surveyed by him.

	1^m	2^m	3^m	4^m	5^m	6^m
Ptolemy	15	48	208	474	217	49
Modern	14	48	152	313	854	210

It appears that Ptolemy on the one hand overestimated the number of brighter stars (down to 4^m) and that on the other hand he missed most of the faint stars – despite his statement that he observed all stars up to the 6th magnitude in so far as they could be sighted [VII, 4; Hei 2, 35]; if this means that he believed himself to have recorded all fixed stars rising above the horizon of Alexandria then he was clearly mistaken. But who will not regard this as an excusable error? As it stands, the stellar catalogue of the Almagest represents a remarkable achievement of ancient astronomy, and many centuries were to pass before astronomers were able to improve upon it.

Concluding Remarks

The establishing of the positions of the fixed stars as a necessary background for the following study of the motions of the planets is the essential feature of Books VII and VIII of the Almagest, and we shall not here give any detailed account of the remaining part of their contents. A special chapter gives a careful description of the Milky Way [VIII, 2; Hei 2, 170] and its passage across the various constellations, but Ptolemy offers no explanation of its physical nature and seems to have no suspicion that it is a collection of faint stars. A following chapter [VIII, 3; Hei 2, 179] describes the construction of a celestial globe upon which the constellations are engraved in the form of

11) Ptolemy's stellar magnitudes have been examined by Lundmark (1926) who compared them with those of al-Ṣūfī (10th century) and Tycho Brahe. He found that they are not sufficiently precise to prove the existence of secular variations in brightness since Antiquity. See also Boll (1916).

simple outlines. The globe is so mounted that it can simulate the diurnal motion of the heavens as seen from different geographical latitudes [VIII, 4; Hei 2, 185]; here the *sphaera recta* and *sphaera obliqua* are carefully described (cf. above page 99) and a number of characteristic positions of the fixed stars relative to the Sun are defined, for instance heliacal risings and settings [VIII, 6; Hei 2, 198]. We shall return to these notions in another connection (page 386). Finally some of the spherical problems dealt with above (page 97 and 110), such as transformation of coordinates from the equatoreal to the ecliptic system, are treated by spherical trigonometrical methods [VIII, 5; Hei 2, 193]. The preoccupation with the particular problem of determining the points of the ecliptic rising, culminating, and setting together with a particular fixed star may be regarded as a reminiscense of the earlier method of defining stellar positions mentioned above in relation to Hipparchus (page 255).

CHAPTER 9

The Superior Planets

Introduction

Having developed the theories of the Sun and Moon with the theory of eclipses as a beautiful corollary, and having described the fixed stars, Ptolemy now turns to the theory of the remaining planets – Saturn, Jupiter, Mars, Mercury, and Venus. They are dealt with in Book IX–XI of the Almagest. The old question of the order of the planets (page 295) is raised at the beginning [IX, 1; Hei 2, 206], where Ptolemy has to admit that there are no objective criteria of planetary distances because of the imperceptible planetary parallaxes. Nevertheless, he thinks it the best course to follow those ancient astronomers who placed the sphere of the Sun in the middle, with Saturn, Jupiter, and Mars above it as superior planets, and Venus, Mercury, and the Moon below it. Mercury and Venus are therefore known as the inferior planets. This has the advantage that the sphere of the Sun divides the planets into two natural groups, the superior having oppositions, and the inferior not.

The problem is now to describe the apparent movement of the five planets in terms of uniform, circular motions, because *only that kind of motion is compatible with the nature of the Divine Beings, whereas irregularity is foreign to them* [IX, 2; Hei 2, 208, cf. page 34 f.]. This task is all the more difficult since nobody has succeeded with it in the past – an indication that the following theories are due to Ptolemy himself, in contrast to the theory of the Sun and Model I of the Moon, where he explicitly states his debt to Hipparchus (see page 122 and 165). The latter seems to have done very little work in planetary theory, apart from determining some fundamental periods [IX, 3; Hei 2, 213], for reasons which Ptolemy tries to guess later (see page 266).

There is no fundamental unity in Ptolemy's theories of the five planets. A common geometrical model would have been preferable, but Mercury in particular refused to conform to the same scheme as the other planets – as we know now because of the exceptionally great eccentricity of its orbit. Only the three superior planets had so much in common that they could be dealt with by means of a common geometrical device, with special numerical parameters adapted to each individual planet. Much of this scheme is found in the theory of Venus too, but the theory of Mercury had to be very different, with a movable deferent like the Moon, and more than one perigee. Although Ptolemy begins with the theory of Mercury we shall here for pedagogical reasons leave the inferior planets to a later chapter and proceed with the superior

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_9,

ones[1]). In particular it will be simpler in this way to explain the various interpolation methods used in numerical calculations of longitudes (see page 292ff).

In the Almagest the superior planets are described as follows:

Material common to the superior planets [IX, 1–6; Hei 2, 206 ff; – X, 6; Hei 2, 316 ff; – XI, 9–13; Hei 2, 426 ff]
The theory of Mars [X, 7–10; Hei 2, 321 ff]
The theory of Jupiter [XI, 1–4; Hei 2, 360 ff]
The theory of Saturn [XI, 5–8; Hei 2, 392 ff]

Since the three theories are developed along very similar lines it is sufficient to study one of them only. Here we shall choose Saturn – a sluggish and not very popular old planet, but with a great influence upon scientists and scholars, with their melancholy temperament dominated by him through his action on the black bile[2]).

The Phenomena of the Superior Planets

The reason why the superior planets can be treated according to the same scheme is not only that they all have fairly small eccentricities, but also that they show the same range of observable phenomena. Many of these phenomena are periodic or almost periodic in so far as successive occurrences are separated by intervals of time varying about constant mean values. In modern astronomy we can distinguish at least five such mean periods (cf. the periods of the Moon's motion page 160).

1) All the planets resemble the Sun and the Moon in having a general motion from West to East among the fixed stars. As shown below (page 264) this direct motion has a varying angular velocity; but it is sufficiently regular to enable us to define both a mean sidereal period of revolution T_f (return to the same fixed star) and a mean tropical period T_t (return to the same point of the ecliptic). The corresponding mean angular velocities ω_f and ω_t are determined by

$$T_f \cdot \omega_f = 360° \quad \text{and} \quad T_t \cdot \omega_t = 360°.$$

2) Since the direct motion is non-uniform, we can also define a mean anomalistic period T_a in which the planet returns to the same velocity, and a corresponding angular velocity ω_a, defined by $T_a \cdot \omega_a = 360°$.

3) The planets exhibit a number of synodic phenomena. Thus each of them abandons its direct motion and is stationary for a short time, after which it becomes retrograde and moves a certain (variable) arc towards the West. At the end of this arc it is stationary again before resuming its direct motion. Furthermore, while the inferior planets both have a limited elongation from the Sun, with which they from

1) Copernicus also began his planetary theory with Saturn, examining Ptolemy's theory (*De Rev.* V, 5) before adjusting it by means of his own observations (ibid. V, 6). A possible reason why Ptolemy started with Mercury is indicated below (note 15).
2) See Klibansky, Saxl, and Panofsky, *Saturn and Melancholy*, London 1964.

time to time are in conjunction, the superior planets can be in opposition as well. This happens to each of them with slightly irregular intervals. Nevertheless we can define a mean synodic period T_s as the average time between two consecutive synodic phenomena of the same kind, for instance oppositions (conjunctions with the Sun are not observable [X, 6; Hei 2, 321]). The corresponding mean angular velocity ω_s is determined by $T_s \cdot \omega_s = 360°$. It is an empirical fact that the synodic period of a given planet is the same regardless of the particular phenomenon used for determining it; for instance, successive oppositions will lead to the same value of T_s as, say, successive shifts from direct to retrograde motion. Another fact is that the phenomena of opposition and retrogradation are coupled in so far as an opposition always takes place in the middle of the retrograde movement.

4) Finally all planets have latitudes varying between a northern maximum and a southern minimum of the same numerical value. This enables us to define a mean draconitic period T_d (return to the same latitude), and a corresponding angular velocity ω_d defined by $T_d \cdot \omega_d = 360°$ and called the mean motion in latitude, an expression which is as misleading here as in the lunar theory (page 165).

This gives a total of no less than five kinematic parameters, viz. the periods T_f, T_t, T_a, T_s, and T_d, and the corresponding angular velocities. For various reasons this number can be reduced to two (within the pale of Ptolemaic astronomy).

First, Ptolemy calculates ecliptic longitudes from the vernal equinox, but not from any fixed star. This means that the sidereal period T_f plays no role. If necessary it can be easily derived from the tropical period T_t. Because of the retrograde motion of the equinoxes T_t is slightly shorter than T_f. With the Ptolemaic rate of precession (page 248) we have

$$\omega_t = \omega_f + 1/100° \tag{9.1}$$

if the mean motions ω_t and ω_f are measured in degrees per year.

Next, Ptolemy believed that the nodal lines of the planetary orbits are fixed relative to the stars (page 357 f.). This means that we have

$$T_d = T_f \quad \text{and} \quad \omega_d = \omega_f \tag{9.2a}$$

Finally, while the anomalistic period T_a in the modern sense is easily defined with reference to the Kepler orbit of the planet, it is a difficult concept to handle in a geocentric theory. In Ptolemaic astronomy it is simply not used with respect to the planets, contrary to what was the case in the theory of the Moon (page 163). Nevertheless, Ptolemy speaks of a 'mean motion in anomaly'; but what he has in mind is always the second anomaly producing the synodic phenomena. Therefore, his 'anomalistic' period is the same as our synodic period, a word which he does not use. This means that we have

$$T_a = T_s \quad \text{and} \quad \omega_a = \omega_s \tag{9.2b}$$

where T_a is the anomalistic period in the Ptolemaic sense. This may appear rather

confusing (cf. note 9, page 139). But the final result is that there are only two fundamental kinematic parameters in the theories of the motions of the planets in longitude, viz. the tropical (or 'zodiacal') period T_t, and the synodic (or 'anomalistic') period $T_s = T_a$.

The First Anomaly of Saturn

The various phenomena listed above are all more or less irregular. Thus the actual periods fluctuate about their mean values, and the retrograde motions are of unequal length and occur at places distributed unevenly along the ecliptic. However, it is possible to distinguish two different classes of irregularities, one of them correlated to the position of the planet relative to the ecliptic, e.g. the motion in longitude and latitude, while the other depends on its position relative to the Sun. This is precisely the distinction which we already know from the lunar theory (page 184). Therefore we shall also in the theories of the planets introduce two fundamental irregularities, or 'anomalies', by means of which all other deviations from uniform motions are to be explained in terms of a suitable geometrical model.

As in the lunar theory, the first anomaly appears in those deviations from uniformity which are correlated to the progress of the planet along the ecliptic, and repeat themselves after a complete revolution. As an example of what that means we may consider the following table, showing the oppositions of Saturn during one complete revolution in the period A.D. 113–142, in the latter part of which Ptolemy made his observations of the planet (see page 273).

Here Column I gives the dates of the oppositions, calculated approximatively by means of Tuckerman's Tables. Column II lists the time interval in days between consecutive oppositions, or the actual synodic periods (see page 263). They are seen to be slowly varying about the mean period $T_s = 378^d$.

Column III contains the ecliptic longitude of Saturn at the various oppositions, given in degrees and decimal fractions of a degree (not in minutes and seconds as usual). Finally Column IV gives the difference of longitude between an opposition and the next following, or Saturn's movement per synodic period in its direct motion along the ecliptic. This velocity varies about the mean value ω_s, and it is precisely this variation of the velocity with the ecliptic longitude which reveals the first anomaly of the planets.

The Second Anomaly

The most spectacular effect of the second anomaly of the planet is its phases of retrograde motion westwards along the ecliptic which from time to time replace the general, direct motion towards the East. As mentioned above (page 263) this takes place around the oppositions which occur near the middle of the period of retrograda-

I Date A.D. t	II Time interval (days) Δt	III Longitude λ	IV Synodic arc Δλ
113 Oct 1		7.03	
114 Oct 15	379	20.97	13.94
115 Oct 29	379	35.15	14.18
116 Nov 12	379	49.44	14.29
117 Nov 26	379	63.80	14.36
118 Dec 10	379	78.09	14.29
119 Dec 24	379	92.22	14.13
121 Jan 6	379	106.15	13.93
122 Jan 20	379	119.73	13.58
123 Feb 2	379	132.95	13.22
124 Feb 15	378	145.85	12.90
125 Feb 27	378	158.40	12.55
126 Mar 12	378	170.66	12.26
127 Mar 24	377	182.65	11.99
128 Apr 5	378	194.40	11.75
129 Apr 17	378	205.85	11.45
130 Apr 29	377	217.18	11.33
131 May 11	377	228.47	11.29
132 May 22	377	239.65	11.18
133 Jun 3	377	250.89	11.24
134 Jun 15	377	262.20	11.31
135 Jun 27	377	273.64	11.44
136 Jul 9	378	285.26	11.62
137 Jul 21	377	297.09	11.83
138 Aug 3	378	309.22	12.13
139 Aug 16	378	321.68	12.46
140 Aug 29	379	334.47	12.79
141 Sep 11	378	347.63	13.16
142 Sep 25	379	1.15	13.52

tion. This proves the connection between the latter phenomenon, and the position of the planet relative to the Sun. Thus this irregularity is essentially of the same kind as the second anomaly of the Moon (the evection, see page 184) although the Moon never exhibits retrograde motions.

The retrograde motions are not the only planetary phenomena coupled to the motion of the Sun. Already Babylonian astronomers distinguished a number of such synodic phenomena. Conjunctions, oppositions, and stationary positions are of this kind, but also others are known. Thus the heliacal rising of the planet is a synodic phenomenon. It appears when the planet, after being invisible about the time of a conjunction, is left so far behind by the fast-moving Sun that it can be seen faintly above the eastern horizon just before sunrise (cf. page 387). The opposite phenomenon

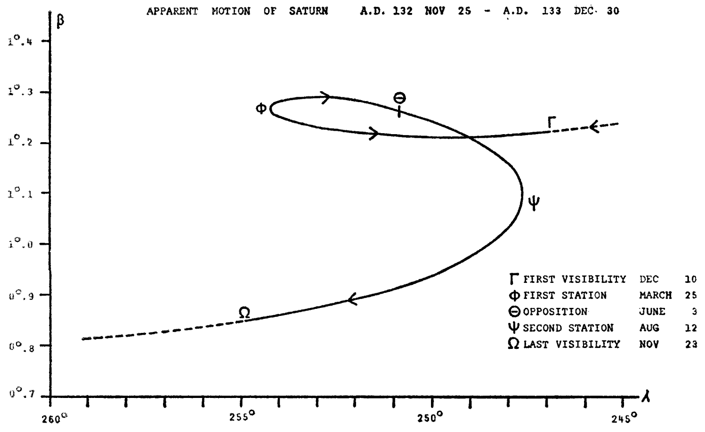


Fig. 9.1

happens when the planet is overtaken by the Sun from the West; its last appearance when it is just visible above the western horizon immediately after sunset is called its heliacal setting. It is, of course, also a synodic phenomenon, and the same is the case of the invisibility period in between.

Synodic phenomena were of particular interest to ancient astronomy, mainly because it is easy to establish the time of their occurrence by means of crude observations without instruments.

The General Purpose of Planetary Theories

We saw above (page 168) that in order to describe the phenomena of the Moon Hipparchus constructed a theory comprising an epicycle riding upon a concentric deferent, and able to account for the first anomaly of the Moon, but not for the evection which was first discovered by Ptolemy. In planetary theory the situation is different in so far as Hipparchus certainly also knew the second anomaly of the five planets, because their retrograde motions were known already to Babylonian astronomers. It is also probable that he realized the impossibility of accounting for the second anomaly by a model constructed on the same general principles as the First Lunar Model. According to Ptolemy [IX, 2; Hei 2, 210] this was the reason why Hipparchus did so little work on the motion of the five planets, but restricted himself to the Sun and the Moon.

Ptolemy on the other hand had shown how a combined epicycle-eccentre model was able to account for the double anomaly of the Moon, so there was no need for him to give up the problem of the planets. On the contrary, he had every reason to try to solve it with a model built along the same lines.

Concerning the specific purpose of such planetary models we notice an important

difference between Hipparchus and Ptolemy on the one hand, and earlier, particularly Babylonian, astronomers on the other. The latter not only based their theories on observations of synodic phenomena, but saw it as the principal aim of astronomy to predict them. Therefore, they tried to find general methods for determining the synodic arcs between consecutive phenomena of the same kind, for example the arcs between consecutive oppositions, listed in Column IV of the table page 265. A relation of the form

$$\lambda_{n+1} = \lambda_n + \Delta\lambda_n$$

will then make it possible to determine the longitude λ_{n+1} of a synodic phenomenon from the longitude λ_n of the previous one, and the synodic arc $\Delta\lambda_n$ between them.

A similar relation

$$t_{n+1} = t_n + \Delta t_n$$

would connect the times of two consecutive synodic events with the intervening synodic period Δt_n.

Now Hipparchus also based his theory of the Moon on synodic phenomena (eclipses), and in the following we shall see Ptolemy proceed in a similar way when he founds his planetary theories mainly (but not exclusively, see page 285) on observations of oppositions; the latter phenomena are analogous to eclipses of the Moon, although the planets are never eclipsed because the shadow of the Earth is too short to reach them. But where the Babylonians would satisfy themselves with, as it were, jumping from one synodic event to the following, both Hipparchus and Ptolemy had a much more general conception of what a planetary theory should be able to do. To them, a planetary theory was not satisfactory and complete unless it was able to determine the position of the planets – including the Sun and the Moon – at any given time t, regardless of whether t is the time of a synodic event or not. In other words, where the Babylonians aimed at relations for $\Delta\lambda_n$ and Δt_n Hipparchus and Ptolemy tried to develop methods leading to expressions (that is, to the equivalent procedures)

$$\lambda = \lambda(t) \quad \text{and} \quad \beta = \beta(t)$$

of both the latitude and the longitude of the planets as what we would call functions of time.

The Choice of a Geometrical Model

To achieve this purpose Ptolemy has at his disposal a number of kinematical devices which are all built upon two fundamental elements. The first is the concentric epicycle model as we know it from Hipparchus' theory of the Moon (the First Lunar Model page 168). The second is the eccentric model used by Hipparchus for his theory of the Sun (page 134). We have also seen how Ptolemy was able to combine both elements in his own lunar theories, and able to analyze the behaviour of a mixed model with an

epicycle riding upon an eccentric circle (page 186). The problem is now if it is possible to account for the motion of the superior planets by a similar combination in such a way that both anomalies are simulated by the model.

As a preliminary to the theory of the Sun, Ptolemy investigated certain general properties of both the eccentric and the epicyclic models. Thus he proved that in a certain case the two models produced exactly the same apparent motion. This led to the equivalent models used in the theory of the Sun (page 137). For any eccentric model (and, of course, its epicyclic equivalent) he also proved the inequality (5.22) which implies that the time used by the planet to pass from the point of minimum to the point of mean velocity is greater than the time from mean to maximum velocity. Since this is an essential property of the eccentric model we can state the following condition: Unless (5.22) is satisfied by a given irregular, circular motion an eccentric model is unable to describe the irregularity.

First, Ptolemy applies this criterion to the second anomaly, that is, the irregularity depending on the relative positions of the planet and the Sun. Without being very specific he refers to *observations of different consecutive positions in the same domain of the ecliptic*, which is said to show that in all the five planets *the time from the maximum to the mean velocity is always longer than the time from the mean to the minimum velocity* [IX, 5; Hei 2, 250]. No actual data are quoted in support of this statement, and the whole matter is dealt with in vague and insufficient terms[3]). In particular it should be noticed that the point of minimum velocity is near to the conjunction where the planet is invisible. However, Ptolemy maintains that (5.22) is not satisfied and concludes accordingly that the second anomaly cannot be accounted for by a model in which the planet moves on an eccentric circle. Consequently this anomaly must be explained in terms of the other model as produced by the motion of the planet on an epicycle.

With respect to the first anomaly, that is the irregularity depending on the ecliptic longitude of the planet, Ptolemy is equally vague. He refers to *arcs on the ecliptic terminated by the same heliacal risings or the same positions*, from which he concludes that now (5.22) is satisfied, since the time from minimum to mean velocity is here longer than the time from mean to maximum velocity [IX, 5; Hei 2, 251].

Now Ptolemy clearly realizes that consequently the first anomaly can be accounted for by both an eccentric and an epicyclic model. Nevertheless, he maintains that the property expressed by (5.22) *is more proper to the eccentric model.* What this means is not clear since the equivalence of the two models has been fully established, and Ptolemy's choice of an eccentric model to account for the first anomaly is somewhat arbitrary. Perhaps it is founded only on economy of thought. We need an epicycle to account for the second anomaly. Consequently we also need a deferent circle to carry the epicycle. But since it is theoretically possible, we can at the same time use this eccentric to account for the first anomaly.

3) There is no doubt that the statement is an argument *post hoc*, offered after Ptolemy for other reasons had decided to relate the second anomaly to the motion on the epicycle; cf. Aaboe 1963 p. 3.

Thus Ptolemy has established the fundamental features of a geometrical model for the motion of the superior planets, viz. that *the motion in longitude is to be understood as the motion of the epicycle centre on the eccentric, and the motion in anomaly as the motion of the planet on the epicycle* [IX, 3; Hei 2, 214]. The problem is now to provide this model with numerical values of the parameters in such a way that it will reproduce or simulate the observed phenomena as precisely as possible.

How this is done will be shown in the following section, but one essential simplification of all the planetary models must be mentioned already at this place. In the lunar theory Ptolemy assumed that the theory of longitude could be developed independently of the theory of latitude (page 168). In his introductory remarks to the planetary theories [IX, 6; Hei 2, 254] he makes the same assumption, although he draws attention to the fact that in the general theory of latitudes in Book XIII he will have to give the plane of the deferent an inclination to that of the ecliptic (as in the lunar theory) and, furthermore, that also the plane of the epicycle will be inclined to that of the deferent (unlike in the lunar theory). But until then all the planetary models used for determining longitudes, and described in Books IX–XII, can be regarded as flat mechanisms having all their circles and other elements situated in the plane of the ecliptic, while all questions of latitude are postponed to Book XIII in which the final, three-dimensional models are introduced.

The Mean Motions of Saturn

The first step towards a theory of the motion of Saturn is the determination of its characteristic periods, or rather the corresponding mean motions (mean angular velocities). At first sight it seems that Ptolemy adheres to the traditional method of establishing period relations of the type

$$Y = P + R \tag{9.3}$$

where Y is the (integer) number of years in which the planet performs R complete revolutions in longitude, and exhibits P synodic phenomena. A relation of this kind is easily found by simple observations (unaided by instruments) over a sufficient number of years. For Saturn the Babylonian astronomers used a relation which we can write in the form[4])

$$256\,T_a = 9\,T_t = 265^a \tag{9.3a}$$

Ptolemy quotes another relation

$$57\,T_a = 2\,T_t = 59^a \tag{9.3b}$$

which is ascribed to Hipparchus [IX, 3; Hei 2, 213]. But the point is that he discards

4) See O. Neugebauer, *Astronomical Cuneiform Texts*, II, 281–283.

it as basis of the following determination of the mean motions, replacing it with the much more sophisticated statement [ibid., Hei 2, 214] that

$$\begin{aligned}&57 \text{ returns in anomaly take place together with}\\ &2 \text{ revolutions} + 1^\circ;43 \text{ in}\\ &59^{a}\ 1\tfrac{3}{4}^{d}\end{aligned} \tag{9.4}$$

This relation is due to Ptolemy himself[5]). The question is, how was it derived? It is so precisely stated that it cannot be the result of direct observations over a relatively short period of about 59 years. The inference is that Ptolemy had observations separated by a much longer span of years at his disposal. That this was so is seen from the table on page 273 where, for instance, the observations P_5 and P_3 are separated by an interval of $t_3 - t_5 = 364^a\ 219^d\ 18^h$. Such observations would enable him to calculate the mean motions directly to a considerable degree of accuracy. Once such mean motions were established it would be easy to correct the crude relation (9.3b) with (9.4) as the result.

If this assumption is true we must draw two conclusions. First, Ptolemy did not base his mean motions on any period relation at all. Second, Ptolemy disguised his actual procedure, presenting the result of a previous calculation in the form (9.4). This may have been for pedagogical reasons to the benefit of readers familiar with period relations. At any rate it enabled him to begin his exposition of the theory with good values of the mean motions.

Since the tropical year is $365^d;14,48$, and 57 returns in anomaly mean a motion of $57 \cdot 360°$ on the epicycle, the daily mean motion in anomaly is

$$\omega_a = \frac{57 \cdot 360°}{59 \cdot 365;14,48 + 1;45} = 0°;57,7,43,41,43,40 \text{ per day} \tag{9.5}$$

This corresponds to a mean anomalistic period

$$T_a = \frac{360°}{\omega_a} \approx 378^d\!\cdot 09 \tag{9.6}$$

The daily mean motion in longitude could be found in exactly the same way as

$$\omega_t = \frac{2 \cdot 360° + 1°;43}{59 \cdot 365;14,48 + 1;45} = 0°;2,0,33,31,28,51 \text{ per day} \tag{9.7}$$

corresponding to a mean tropical period equal to

$$T_t = \frac{360°}{\omega_t} \approx 10743^d \approx 29^a.\,41 \tag{9.8}$$

5) In the *Hypotheses* (I, 17; *Opera minora* p. 79) this relation is changed. Here Ptolemy states that Saturn performs 313 anomalistic revolutions minus 0°;12,26,19,14,25,48 in 324 tropical years, or 324 Egyptian years and 83 days. Similar relations comprising long periods and ignoring the number of revolutions in longitude are given for the other planets.

But in order to avoid this tedious calculation Ptolemy proceeds in a different way, using a relation discussed later [page 284, cf. X, 6; Hei 2, 318] and stating that

$$\lambda_{m\odot} = \lambda_m + a_m \tag{9.9}$$

where

$\lambda_{m\odot}$ = mean longitude of the Sun
λ_m = mean longitude of the planet
a_m = mean anomaly of the planet

Differentiating, and using $\dot{\lambda} = \omega$, we get

$$\omega_{\odot} = \omega_t + \omega_a \tag{9.10}$$

so that ω_t can be found by means of (5.1) and (9.5)

The relation (9.10) is often stated in another way. Denoting one tropical year by T, and using (9.6) and (9.8) we obtain the formula

$$\frac{1}{T} = \frac{1}{T_a} + \frac{1}{T_t} \tag{9.11}$$

which is true for the three superior planets. A similar relation for the inferior planets is given later (page 296).

As usual, Ptolemy also computes the mean motions per hour, per month (30^d), per Egyptian year (365^d) and for 18-year periods. These parameters are the basis of the *Saturn tables* found in the Almagest [IX, 4; Hei 2, 220] along with similar tables for the four remaining planets. By means of these tables it is easy to calculate the mean longitude at a given time by means of the usual formula analogous to (6.2)

$$\lambda_m(t) = \lambda_m(t_0) + \omega_t(t - t_0) \tag{9.12}$$

and also the mean anomaly (cf. 6.44)

$$a_m(t) = a_m(t_0) + \omega_a)t - t_0) \tag{9.13}$$

provided that the *radices*, or epochal values $\lambda_m(t_0)$ and $a_m(t_0)$, are known. They will be determined later (page 290).

The Empirical Data of the Saturn Theory

We must now begin to construct a geometrical model describing Saturn's motion and containing both an epicycle and an eccentric deferent, the parameters of which must be deduced from observations. This is, in a way, the same problem as in the lunar theory, but it presents us with a new difficulty. The theory of the Moon was developed step by step from the first, or Hipparchian model in which only one anomaly was taken into account. But in the case of the superior planets we know already at the beginning that there are two anomalies, i.e. two independent deviations from uniform circular motions. How can we know – before the model has been finished – how each

component affects the position of the planet? Or, in other words, how can we get information on each separate component out of any observation of their combined effect?

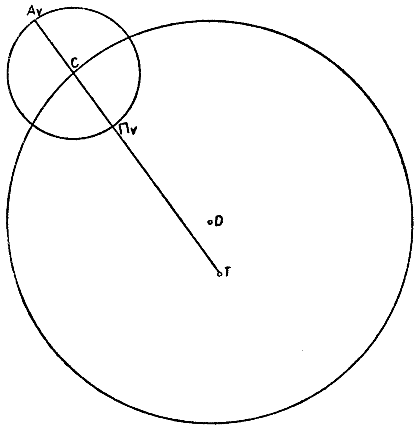


Fig. 9.2

The answer is that this is impossible – unless we are able to single out a particular class of observations which depend only on one of the two components. Here the observable fact that oppositions occur in the middle of the retrograde arc comes to our help. It appears from Figure 9.2 that in the middle of the retrograde motion the planet will be either at the apogee A_V or the perigee Π_V of the epicycle, depending on whether the motion on the epicycle is itself indirect (in which case the opposition takes place at A_V) or direct (opposition at Π_V).

This means that at opposition the planet has the same ecliptic longitude as the epicycle centre C, the motion of which is affected by one anomaly only (the first, or eccentric anomaly related to the motion along the ecliptic). In other words, observations of oppositions give information on the motion of the epicycle centre and should therefore lead to the parameters of the eccentric circle. We remember that the lunar theory too was founded upon oppositions (lunar eclipses), but for a different reason, viz. to get observations unaffected by parallax (page 161); this does not trouble us here (cf. page 261).

The amazingly few observations upon which the Saturn theory is founded are listed in the following table. It is seen that they are all made by Ptolemy himself, apart from P_5 which is by an unknown Babylonian astronomer. The table is arranged like the Table of Lunar Eclipses and the contents of the various columns is the same as explained on page 170. The first three observations are of oppositions. P_1 was measured directly with the *astrolabon*, while P_2 and P_3 occurred during daytime so that their longitudes had to be derived by interpolation between a preceding and a succeeding night observation of which no particulars are given. P_4 is a simple measurement of longitude made with the *astrolabon*. Finally the Babylonian observation P_5 is stated

in the words that Saturn stood 2 digits below the southern shoulder of Virgo (γ Virginis), which enables Ptolemy to determine its longitude by taking the precession of the star into account.

Saturn Observations in the Almagest

I	II	III	IV	V	VI
P_1	Hadrian 11 Pachon 7/8 Alexandria	6^h	181°;13	A.D. 127 March 26	55
P_2	Hadrian 17 Epiphi 18 Alexandria	4^h	249°;40	A.D. 133 June 3	65
P_3	Hadrian 20 Mesore 24 Alexandria	Noon	284°;14	A.D. 136 July 8	74
P_4	Antoninus 2 Mechir 6/7 Alexandria	8^h	309°;4	A.D. 138 Dec 22	81
P_5	Nabonassar 519 Tybi 14 Mesopotamia	6^h	[159°;30]	B.C. 229 March 1	29

It is important to notice that these oppositions are not defined with respect to the true, but to the mean Sun. Thus the longitudes in Column IV must be understood as $180° + \lambda_{m\odot}$ where $\lambda_{m\odot}$ is given by (5.2).

The Equant – First Approximation

We are now prepared to tackle the problem of determining the eccentric circle upon which the epicycle centre C (see figure 9.3) is supposed to move with the constant angular velocity ω_t in such a way that the first anomaly is produced [XI, 5; Hei 2, 392]. This circle has many names. Ptolemy spoke of the *circle producing the anomaly*, or the *circle of uniform motion.* Latin astronomers usually called it the *circulus equans.* In the following we shall simply call it the equant. In the figure it has its centre at E and its apogee at A. As in the theories of the Sun and the Moon its radius has the arbitrary value $R = 60^p$, where 1^p is a unit peculiar to the planet in question. The circle is defined by the longitude λ_a of its apogee A, and the eccentricity TE which we for reasons explained later denote by 2e. The three oppositions are observed on the ecliptic at the points P_1^*, P_2^*, P_3^* with the longitudes λ_1, λ_2, λ_3. These points are projections from T of the three successive positions of the epicycle centre C_1, C_2, C_3.

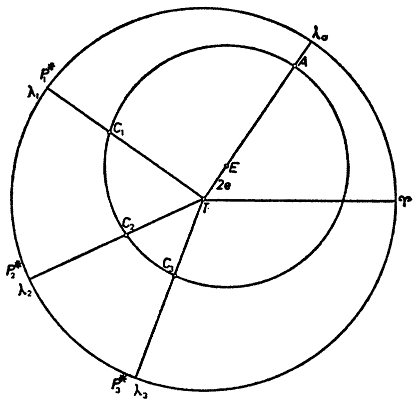


Fig. 9.3

From the given empirical data we can easily find the differences of true longitude between the three oppositions, and also the corresponding time intervals. Entering with the latter in the table of mean motion in longitude we can also find the differences of mean motion in longitude, that is, the arcs of the eccentric circle between the oppositions. The result is for the times

$$t_2 - t_1 = 6^a\ 70^d\ 22^h$$
$$t_3 - t_2 = 3^a\ 35^d\ 20^h$$

and for the true longitudes

$$\lambda_2 - \lambda_1 = 68°;27 = P_1{}^*\ P_2{}^*$$
$$\lambda_3 - \lambda_2 = 34°;34 = P_2{}^*\ P_3{}^*$$

and finally for the mean longitudes

$$\lambda_{m2} - \lambda_{m1} = 75°;43 = \sphericalangle\ C_1E\ C_2$$
$$\lambda_{m3} - \lambda_{m2} = 37°;52 = \sphericalangle\ C_2E\ C_3$$

Here as elsewhere Ptolemy does not use longitudes as standard coordinates, preferring the differences

$$c \ = \lambda \ - \lambda_a \qquad (9.14\ a)$$
$$c_m = \lambda_m - \lambda_a \qquad (9.14\ b)$$

which he sometimes calls the 'zodiacal anomaly'. To avoid confusion we shall adopt the standard terms of the Latin astronomers and call c and c_m the t r u e and the m e a n c e n t r u m respectively (cf. the c e n t r u m in the lunar theory page 193). The transformation (9.14) means no more than that the centrum-coordinates are measured from the apogee, whereas longitudes are reckoned from the vernal equinox. From (9.14 b) and (9.12) we can derive the expression

$$c_m(t) = \lambda_m(t_0) + \omega_t(t - t_0) - \lambda_a(t) \qquad (9.15\ a)$$

If the longitude of the apogee is constant (cf. page 287) we have $\lambda_a(t) = \lambda_a(t_0)$, and the expression can be written

$$c_m(t) = c_m(t_0) + \omega_t(t - t_0) \tag{9.15 b}$$

which gives the mean centrum as a linear function of time. Since (9.15 b) contains the same angular velocity as (9.12) mean centrum differences can be found by the tables of mean motion in longitude. It follows that the differences above can be expressed as

$$\begin{aligned} c_2 - c_1 &= 68°;27 \\ c_3 - c_2 &= 34°;34 \end{aligned} \tag{9.16}$$

and

$$\begin{aligned} c_{m2} - c_{m1} &= 75°;43 \\ c_{m3} - c_{m2} &= 37°;52 \end{aligned} \tag{9.17}$$

provided that the value of λ_a has not changed during the period of time from the first to the third opposition.

We are now faced with the problem of determining the apogee λ_a and the eccentricity 2e of an eccentric circle upon which two consecutive arcs of 75°;43 and 37°;52 are projected from the centre T of the Earth upon the ecliptic as arcs of 68°:27 and 34°;34 respectively.

This is obviously a special case of the general Hipparchian problem formulated on page 173, and there solved by Ptolemy when determining the parameters of the epicycle of the Moon. In the latter case the centre of the Earth lay outside the circle to be determined, while in the theories of the superior planets it lies inside it. But this is immaterial to Ptolemy's solution (page 174), which accordingly must apply also to the present case. Actually (9.16) are analogous to the relations (6.10) of the lunar theory, while (6.9) corresponds to

$$(\lambda_2 - \lambda_1) - (\lambda_{2m} - \lambda_{1m}) = -7°;16 \tag{9.18}$$
$$(\lambda_3 - \lambda_2) - (\lambda_{3m} - \lambda_{2m}) = -3°;18 \tag{9.19}$$

in the case of Saturn. These analogies illustrate the formal agreement between the two problems.

Here we shall not carry out the actual calculations involved in the solution, but only quote the resulting parameters found in the Almagest. They are the eccentricity

$$2e = 7^p;8$$

and the position of the apogee of the eccentric relative to the second opposition

$$c_{m2} = \lambda_{m2} - \lambda_a = 19°;51 \tag{9.20}$$

which latter value means that the second opposition occurs 19°;51 East of the apogee, measured upon the eccentric circle. The parameter

$$\lambda_a = 233°;0 \tag{9.21}$$

that is, the position of the apogee relative to the vernal equinox measured upon the

ecliptic, cannot be found without a separate trigonometrical calculation (see page 282), which Ptolemy does not perform until he has changed the model in a very significant way.

The Deferent

The equant being determined, one would have expected Ptolemy to continue the development of the Saturn theory by determining the radius of the epicycle, and the *radices* of the mean longitude and mean anomaly at the standard epoch. This would have resulted in a complete model which could have been tested against observations, and modified if it failed to reproduce them with sufficient accuracy. Such was Ptolemy's procedure in the lunar theory (page 177), but in the theories of the planets he follows a different course. In fact, he introduces a significant modification of the model before it is completed. Before determining the epicycle he discards the equant as a carrier of the epicycle centre, replacing it by another circle which we shall call the *circulus deferens*, or simply the deferent[6]).

Why this is so is not sufficiently explained in the Almagest. Already in his introductory remarks on planetary theories in general Ptolemy simply states that he has *found that the epicycle centres revolve upon circles of the same size as the eccentric circles producing the anomaly, but not with the same centres* [IX, 5; Hei 2, 252]. This is about all he tells us about it. There is no explanation at all of how he found this. On the contrary, he immediately continues with a rule for determining the deferent centre D (figure 9.4) as the middle point between the equant centre E and the centre T of the Earth. Thus we have the eccentricity of the deferent TD = e. This is the reason why the eccentricity of the equant was denoted by 2e (page 273).

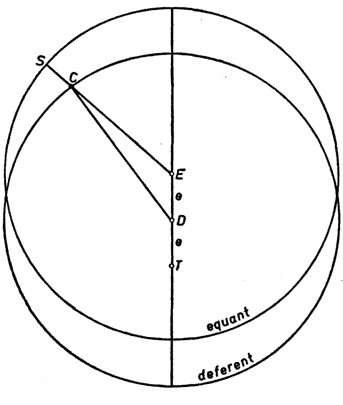


Fig. 9.4

6) Thus it is misleading when Dreyer (1906, p. 197) believes the equant centre to be introduced later than the deferent centre, or "the centre of distances".

The Bisection of the Eccentricity

The bisection of the eccentricity (to employ an often used term) means that now the model is provided with two eccentric circles of equal radii $R = 60^p$, but different centres. The first is the equant (centre E and eccentricity 2e) which is discarded as the locus of the epicycle centre C, but retained as the circle of uniform motion, in so far as a certain point S is moving upon its periphery with the constant angular velocity ω_t. The second is the deferent (centre D and eccentricity e) upon which the epicycle centre C is determined as the point of intersection of the deferent circle and the rotating radius ES of the equant [IX, 6; Hei 2, 253].

This kinematical device has the obvious consequence that the motion of C is still circular around D, but uneven as seen from D. In other words we are faced with a manifest violation of the fundamental assumption of earlier Greek astronomy that all circular motions used as components in the various theoretical models must be uniform with respect to their own centres in order to agree with *the nature of the Divine Beings* (see page 35). To make them uniform only with respect in the centre of another circle must have been as revolting to Hellenistic astronomers as to Greek philosophers in general. An echo of this scandal is heard as late as Copernicus, who after many centuries admitted that it was one of the incentives in his search for an alternative planetary theory in which the old dogma was strictly upheld[7]). After all, Ptolemy's kinematical heresy was without doubt a highly provocative piece of sophistication which only very important reasons could justify.

Unfortunately Ptolemy did not reveal these reasons. Perhaps his silence is due to an understandable desire not to make his new planetary theory too insupportable to his fellow astronomers and to contemporary philosophy.

For the history of astronomy this silence is a deplorable fact since it leaves us to make our own more or less plausible conjectures. But one inference remains uncontradicted: since Ptolemy states that he discarded the equant as a result of observations, his planetary theory must have been so far developed that he was able to check it by means of observations. Now the epicycle centre is invisible, and its exact position can be ascertained only on those rare occasions when the planet is in opposition to the Sun. But since the parameters of the equant were found by means of oppositions, it is necessary to suppose that Ptolemy was able to check his theory by means of observations of another kind.

Therefore, we must conclude that before discarding the equant as carrier of the epicycle centre Ptolemy must have had at his disposal a finished model, complete with a prosthaphairesis-table (or an equivalent procedure), and with an epicycle riding on the equant circle determined above. Thus we can infer the existence of a First Planetary Model analogous to the First Lunar Model (page 167). In a similar

7) In *De Revolutionibus* Copernicus returns several times to this break with the established opinion of how uniform, circular motion is to be understood; see the preface (fol. iii verso) where his criticism is stated in general terms, and his remarks on the Ptolemaic theories of the Moon (IV, 2) and Mercury (V, 25). – A later author like Tannery (1893, p. 256) still speaks of Ptolemy's little heresy as *le vice fondamental* of his system.

way the new model with the bisected eccentricity and the deferent introduced above as carrier of the epicycle may be denoted as Ptolemy's Second Planetary Model.

In these terms we can now ask the two crucial questions: 1) why was the First Model so unsatisfactory that it had to be replaced by the Second Model? and 2) why was the eccentricity of the deferent chosen as precisely one half of the eccentricity of the equant?

As already mentioned, the Almagest gives no explicit answer to the first question[8]). But if we look for observable differences between the First and Second model it is obvious that the apparent size of the epicycle will be affected by the transition from the one model to the other. Actually, at the apogee of the eccentric circle the epicycle will be drawn nearer to the Earth in the Second Model, and accordingly look larger. At the perigee the opposite will be true. Therefore, it is quite possible that Ptolemy might have found the First Model unable to account for the size of the epicycle. We remember that he made the transition from the First to the Second Lunar Model for similar reasons (page 184).

The size of the epicycle will influence several phenomena, one of which is the length of the retrograde arc (page 343), which can be found directly by observations. Another is the latitude of the planet, according to the theory developed in Book XIII (page 355). In fact the hypothesis that Ptolemy had trouble with the latitudes derived from his First Model was put forward by Kepler, who made an ingenious and detailed attempt to solve the riddle in connection with his own investigations on the motion of Mars[9]).

Finally it is possible that Ptolemy was worried more by the distance than by the size of the epicycle. This means that the First Model might have been unable to account for the velocity of the planet in different parts of the eccentric circle, particularly the velocity at the oppositions[10]. Of course, this guess is just as hypothetical as

8) Robert Small wrote (1804, pp. 55–56) that *as Ptolemy gives no account of the means by which this was discovered, nor of the observations from which it was infered, his assuming it has justly excited the wonder of all astronomers. The greater part believed him to have assumed it merely from conjecture, and not to have derived it, as Kepler more generously supposed, from any observations.* Small himself suggested that *there seems some reason for thinking, that it came to him by tradition, from the more ancient astronomy of the east.* This is without any support whatever and only reflects the general ignorance of eastern astronomy at the time when Small wrote his survey.
9) The gist of Kepler's long and complicated argument is the following: Assuming the bisection of the eccentricity Ptolemy had shown how λ_a and $e = TD = DE$ could be found from three oppositions. Discarding this assumption Kepler investigated a model for Mars – later called the *hypothesis vicaria* – in which λ_a, the total eccentricity TE and the ratio TE:DE could be determined from four oppositions (see *Astronomia Nova* (De motibus stellae Martis) II, 16, *Opera omnia* III, p. 151 ff.). Using four of Tycho Brahe's observations, Kepler found (p. 168) a total eccentricity TE = 0.18564 divided in the ratio TD:DE = 0.11332:0.07232. This model was found to reproduce the longitudes of 12 oppositions to within about 0°;2 (II, 18; pp. 171 ff.), but was unable to reproduce the latitudes of the planet. However, Kepler proved that the latitudes could be accounted for if the total eccentricity was bisected as TD = DE = 0.09282; but in this case the errors in longitude rose to 0°;8 (II, 19; p. 177). Without any evidence whatever Kepler surmised that Ptolemy might have found himself in a similar situation, accepting the bisection of the eccentricity in order to save the latitudes, and tolerating an error of 0°;8 in longitude (assuming that the was aware of it) as acceptable within the general accuracy of his observations. A similar course was not open to Kepler because of the greater accuracy of Tycho's observations, and he had to discard both the *hypothesis vicaria*, and the bisection of the eccentricity.
10) This explanation was put forward by Dreyer (1906, p. 197) who also gave a good account of Kepler's explanation (ibid., pp. 384 ff.).

the preceding ones, and it may well be that we shall never be able to find a definite answer to the first question, and that Ptolemy's own silence must reign forever.

Concerning question 2 we are, perhaps, a little better off, since it is possible to understand the bisection of the eccentricity in at least two different ways. One hypothesis is that it is the first step in an iterative procedure. Ptolemy does not say that in so many words; he only states that he found that the centre C of the deferent had to be in the middle between the centre T of the Earth and the centre E of the equant [IX, 5; Hei 2, 252]. But if one considers what he actually does the matter becomes more clear. In fact, when he discovered whatever it was that went wrong with the first model he must have asked himself the question: where could one place C so that the defect of the model was remedied?

In other words Ptolemy must have been led to consider the original model with the eccentricity $e_1 = 2e = 7^p;8$ as a first approximation to a satisfactory theory. His next step would then be to define a second approximation with a different eccentricity e_2 and test it against observational data. He may even have envisaged a third approximation. But in an iterative procedure of this kind it would be natural to begin by putting $e_2 = e$. Any other choice would have been more arbitrary.

Another hypothesis explaining the bisection of the eccentricity will be discussed later in connection with the theory of Venus (page 306).

The Equation of Centre

It is obvious that the introduction of the deferent must invalidate the previous determination of the equant circle. In Figure 9.5 it is seen that when the epicycle centre is displaced from S to C the planet will at oppositions lie in the direction of TC, and not in the direction of TS as before. In other words, what we know from observation of oppositions is the longitude of the point C, not that of S. This means that we must determine a correction equal to the angle $\delta = STC$.

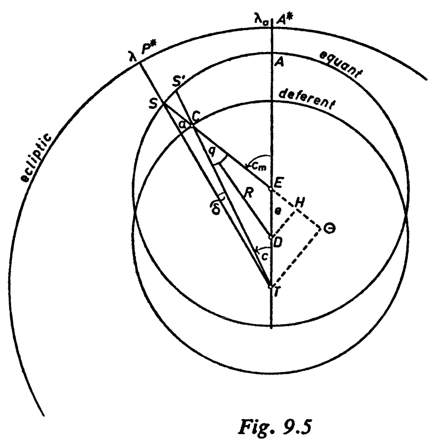


Fig. 9.5

We shall first find an expression for the distance TC = $\rho(c_m)$ of the centre of the epicycle from the centre of the Earth, analogous to the formula (6.46) of the lunar theory. This distance will be a function of the mean centrum c_m defined by (9.15). Constructing the points H and Θ as shown in Figure 9.5 we find

$$T\Theta = 2e \sin c_m = 2 \cdot DH$$
$$E\Theta = 2e \cos c_m = 2 \cdot EH = 2 \cdot H\Theta$$

where e has the value implied in (9.19). By Pythagoras we have

$$CH = \sqrt{R^2 - DH^2} = \sqrt{R^2 - (e \sin c_m)^2}$$

We also have

$$C\Theta = CH + H\Theta = CH + e \cos c_m$$

We can now find $\rho(c_m)$ from the triangle CTΘ

$$\rho(c_m) = \{[\sqrt{R^2 - (e \sin c_m)^2} + e \cos c_m]^2 + (2e \sin c_m)^2\}^{\frac{1}{2}} \tag{9.22}$$

The angle TCE = q can now be found. It is the angle under which the line TE is seen from the epicycle centre, and is called the *equatio centri*, or the equation of centre. It can be determined from the triangle TCΘ by

$$\sin q = \frac{T\Theta}{TC}$$

or

$$\sin q(c_m) = -\frac{2e \sin c_m}{\rho(c_m)} \tag{9.23}$$

Accordingly it is a function of the mean centrum c_m, just as the analogous variable (6.48) in the lunar theory.

It follows from the triangle TCE in Figure 9.5, that the equation of centre connects the mean and true centrum by the relation

$$c = c_m(t) + q(c_m) \tag{9.24}$$

Since $c_m > c$ in the interval $0° < c_m < 180°$, but $c_m < c$ in the interval $180° < c_m < 360°$ it follows that (9.23) gives $q(c_m)$ with the correct sign. Adding the longitude of the apogee λ_a on both sides of (9.24) we obtain an expression

$$\lambda_c = \lambda_m(t) + q(c_m) \tag{9.25}$$

where λ_c is the true longitude of the epicycle centre C; in Latin astronomy this coordinate was sometimes called the *verus motus epicicli.*

The relation (9.25) shows how the linear function of time $\lambda_m(t)$ is transformed into the non-linear function λ_c through the addition of the equation of the centre. Thus the epicycle centre will perform an uneven motion on the deferent in accordance with the first anomaly.

By a computation similar to that leading to (9.22) Ptolemy determines the distance TS, and then the angle TSΘ = α from

$$\sin \alpha = \frac{T\Theta}{TS} = \frac{2e \sin c_m}{TS} \tag{9.26}$$

It is seen that α would have been the equation of the centre if C had been moving on the equant instead of on the deferent. In other words, α is the equation of centre of the First Model (see page 277). Then the angle

$$\delta = q - \alpha \tag{9.27}$$

is the correction we sought. We shall provide it with a sign according to the relation

$$\lambda(\text{observed}) + \delta = \lambda(\text{corrected}) \tag{9.28}$$

Now Ptolemy is able to calculate δ for each of the three oppositions P_1, P_2, P_3. He finds the following results:

I	II λ (obs)	III δ	IV λ (corr)
P_1........	181°;13	−0°;9	181°;4
P_2........	249°;40	+0;6	249°;46
P_3.......	288°;14	+0°;10	284°;24

Here Column II contains the observed longitudes $\lambda(P^*)$ of the planet at opposition (page 273), equal to the longitudes $\lambda(S)$ of the point S on the equant in the first approximation. Column IV contains the corrected values, that is the longitudes of the point C on the deferent equal to the longitudes $\lambda(S')$ of the point S′ on the equant (see figure 9.5).

In this way we have established a correspondence between the three points of opposition P* on the ecliptic, and three points S′ on the equant. The latter have the longitude differences

$$\begin{aligned} \lambda_2 - \lambda_1 &= 68°;42 \\ \lambda_3 - \lambda_2 &= 34°;38 \end{aligned} \tag{9.29}$$

These are also the differences of the true centrum

$$\begin{aligned} c_2 - c_1 &= 68°;42 \\ c_3 - c_2 &= 34°;38 \end{aligned} \tag{9.30}$$

which are the corrected equivalent to (9.16). Since the differences of the mean centrum c_m depend only on the times of the oppositions, they remain unchanged and are given by (9.17).

The equant in the second approximation can now be determined as a circle upon which two consecutive arcs of 75°;43 and 37°;52 are projected from T upon the ecliptic as arcs of 68°;42 and 34°;48 respectively (cf. page 275). The procedure is exactly the same as before and Ptolemy omits the actual calculations. The results are listed here together with the corresponding values from the first approximation.

	I	II
2e........	$7^p;8$	$6^p;50$
c_{m1}.......	304°;8	302°;55
c_{m2}.......	19°;51	18°;38
c_{m3}......	57°;43	56°;43

It follows that the distance of the deferent centre D from the centre of the Earth in the second approximation is changed to[11])

$$e = 3^p;25 \tag{9.31}$$

No Third Approximation

The question is if this second approximation is satisfactory, or if we have to proceed with a third approximation. To decide this question Ptolemy checks whether the model in the second approximation is able to give the observed positions of the planets at the times t_1, t_2, t_3 given in the list of empirical data page 273. Therefore Ptolemy calculates the equation of centre and uses (9.24) to find the true centrum at each of the three oppositions. This gives the following results

	c_m	$q(c_m)$	c
P_1........	302°;55	5°;18	308°;13
P_2........	18°;38	1°;58	16°;40
P_3.......	56°;43	5°;16	51°;14

the actual calculations being left out by Ptolemy himself.

This gives the differences

$$\begin{aligned} c_2 - c_1 &= 68°;27 \\ c_3 - c_2 &= 34°;34 \end{aligned} \tag{9.32}$$

11) In decimal notation Ptolemy's result is

$e = 0.0569444$

Calculating the same parameter from a Kepler orbit Czwalina (1958, p. 299) found

$e = 0.0568451.$

which are precisely identical with the observed differences (9.16). Ptolemy concludes accordingly that the second approximation is able to reproduce the actual observations, and that no further steps are necessary [XI, 5; Hei 2, 407]. Therefore Ptolemy finishes this section by computing the longitude λ_a of the apogee, which he did not carry through the successive approximations. We have from (9.14)

$$\lambda_a = \lambda - c$$

which by (9.24) gives

$$\lambda_a = \lambda - c_m(t) - q(c_m) \tag{9.33}$$

Using the values corresponding to the first observation P_1 this gives (9.21), i.e. $\lambda_a = 233°;0$[12]). Also in the theory of Jupiter the second approximation proves to be sufficient [XI, 1; Hei 2, 375] while a third approximation is necessary in the case of Mars[13]), because of the large eccentricity of this planet [X, 7; Hei 2, 339].

The Motion on the Epicycle

The Saturn theory has now been so far developed that we are able to describe the motion of the epicycle centre C on the deferent, taking the first anomaly into account through the equation of the centre (9.23). The next step must be to provide the model with an epicycle of a proper size and direction of revolution in such a way that the second anomaly is reproduced.

Let us first consider the direction in which the planet moves upon the epicycle. We have seen above (page 268) that from a theoretical point of view it is possible to choose either a direct or an indirect motion. However, the choice is not completely arbitrary. 1) The fact that the planet has its maximum apparent brightness around the opposition seems to indicate that here it has its minimum distance from the Earth; accordingly oppositions must occur near the true perigee Π_v of the epicycle (see figure 9.2). 2) Another fact is that oppositions occur in the middle of the retrograde periods (see page 263).

From 1) and 2) follows that the planet is retrograde around the perigee of the epicycle. But this is the same as to say that its motion on the epicycle is direct, contrary to what the case was in the theory of the Moon[14]).

12) Determining the elliptic Kepler orbit from Ptolemy's observations Czwalina (1958 p. 299) found the longitude of the apogee to be $\lambda_a = 180° + 53°;3$. This compares well with Ptolemy's result.
13) Concerning the Ptolemaic theory of Mars, see Max Caspar's introduction to the German translation of Kepler's *Neue Astronomie*, München-Berlin, 1929, pp. 9*–12*.
14) There is some evidence that Greek astronomers investigated also models in which the planet has a retrograde motion on the epicycle (as in the lunar theory). A model of this kind is implied in *Papyrus Michigan* 149 dating from the 2nd century A.D. It reminds one of the models vaguely described by Pliny, *Hist. nat.* II, 12–14. This type of models has been examined by Aaboe (1963) who proved that it is impossible to provide them with such parameters that retrogradations will occur for all the five planets.

To describe the motion of P on the epicycle it will be convenient to introduce two new variables. They are called the mean and true anomaly, but in order to avoid the confusing word anomaly we shall call them the mean and true argument after the Latin terms *medium argumentum* and *verum argumentum*. Formally speaking the epicyclic variables are defined here in the same way as in the lunar theory (page 194).

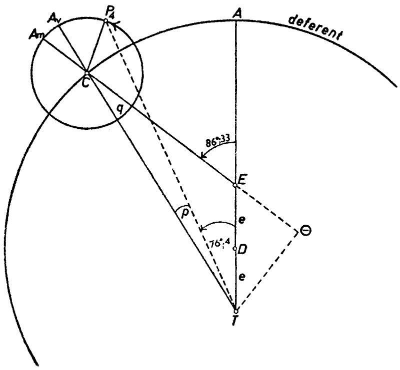


Fig. 9.6

Thus the mean argument $a_m(t)$ is the angular distance of P from the mean apogee A_m (or the *aux media*) of the epicycle (see Figure 9.6), that is the point with the maximum distance from the equant centre E. It is equal to the arc $A_mP = a_m(t)$ on the epicycle, reckoned according to the direction of motion, from A_m to P. This coordinate $a_m(t)$ is supposed to be a linear function of time given by (9.13). Its rate of increase is the mean anomalistic angular velocity ω_a.

The true argument a_v is the angular distance $A_vP = a_v$ of P from the true apogee A_v, that is from the point of the epicycle with the maximum distance from the centre of the Earth T. It follows from Figure 9.6 that a_m and a_v are connected by the relation

$$a_v = a_m(t) - q(c_m) \tag{9.34}$$

which is analogous to (6.45) in the lunar theory. That $q(c_m)$ has the opposite sign is due to the fact that the Moon has an indirect, but the planet a direct motion on the epicycle.

Since the theory was founded on the presupposition (page 269) that the Ptolemaic 'mean motion in anomaly' is identifiable with the motion on the epicycle, the latter motion must be coupled to the motion of the Sun in such a way that the second anomaly is produced by the model. This coupling is effected by the relation

$$\lambda_{m\odot} = \lambda_m + a_m \tag{9.9}$$

the derivative of which we used (page 271) for finding the value of ω_t of the planet. This relation is discussed in a particular chapter [X, 6; Hei 2, 318], where it appears as a fundamental postulate of the theories of all the superior planets. It has a simple

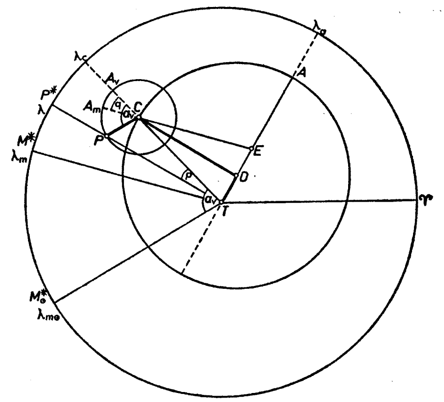


Fig. 9.7

geometrical interpretation, viz. that the epicycle radius CP to the instantaneous position P of the planet is parallel to the line $TM_{\odot}{}^{*}$ from the Earth to the ecliptic mean Sun. This appears from Figure 9.7, which shows that when CP is parallel to $TM_{\odot}{}^{*}$ we have

$$\lambda_{m\odot} = \lambda_c + a_v$$

which gives (9.9) by means of (9.25) and (9.34). That the coupling expressed by (9.9) is a necessary feature of the models of both Saturn, Jupiter, and Mars is understandable from a Copernican point of view since the motion of the planet on the epicycle in the geocentric theory is a simple reflection of the annual motion of the Earth around the Sun in the heliocentric arrangement.

The Size of the Epicycle

Among the geometrical parameters of the Saturn model only the radius r = CP of the epicycle remains undetermined. Of course it is impossible to find r from observations of oppositions, since here the radius CP points directly towards the Earth (page 283). Therefore Ptolemy made the observation P_4 (page 273) which showed (by means of the astrolabon) that at the time t_4 Saturn had the true longitude

$$\lambda(t_4) = 309°;4$$

from which follows the true centrum (the angle ATP_4)

$$\lambda(t_4) - \lambda_a = 76°;4$$

The time difference between P_3 and P_4 is

$$t_4 - t_3 = 2^a 167^d 8^h$$

By putting $t_0 = t_3$ in (9.15 b) we can now find the mean centrum

$$c_m(t_4) = 86°;33$$

because $c_m(t_3)$ is known from (9.31). The latter value enables us to place the epicycle centre correctly on the deferent, as shown in Figure 9.6. To find the position of the planet itself on the epicycle we must use (9.23) to determine

$$q(t_4) = 6°;29$$

and also (9.13) to calculate

$$a_m(t_4) = 309°;8$$

From these values r can be found. In Figure 9.6 we first consider the triangle TCE which has c_m as an exterior angle, from which follows

$$c_m = q + c + (-p)$$

where $(-p)$ is the prosthaphairesis-angle under which the radius CP is seen from the Earth (cf. later page 288 where the sign of p is explained). With the given values we find

$$(-p) = 4°;0$$

In the triangle TCP we have the

$$\text{angle PCT} = a_m - 180° + q = 135°;37$$

and the

$$\text{angle CPT} = 180° - (135°;37 + 4°;0) = 40°;23$$

while the side

$$TC = \rho(c_m) = 60^p;29$$

is calculated from (9.22). Applying the sine relation to this triangle we find

$$r = \frac{\sin 4°;0 \cdot 60^p;29}{\sin 40°;23}$$

or

$$r = 6^p;30 \tag{9.35}$$

The Complete Model

We can now describe the complete Saturn model (Figure 9.7) geometrically by the vector relation

$$\overrightarrow{TP} = \overrightarrow{TD} + \overrightarrow{DC} + \overrightarrow{CP} \tag{9.36}$$

which corresponds to the geometrical construction of the position of the planet explained in the Almagest [XI, 9; Hei 2, 426]. The position vector $\overrightarrow{TP}$ from the Earth to the planet is here resolved into three components.

First the eccentricity vector $\overrightarrow{TD}$ has the length $e = 3^p;25$ and points towards the apogee A of the deferent. Unlike the apogee of the Sun, A is supposed[15]) to be a fixed point relative to the fixed stars [IX, 6; Hei 2, 254]. Its longitude λ_a will accordingly increase by 1° per 100^a because of the precession of the equinoxes. Thus we have

$$\lambda_a(t) = \lambda_a(t_0) + 1° \cdot (t - t_0) \tag{9.37}$$

where $(t - t_0)$ is reckoned in centuries.

The deferent vector $\overrightarrow{DC}$ has a constant length of $R = 60^p;0$. It rotates around D describing the deferent circle as locus of the epicycle centre C. As we have seen, this rotation is not uniform as seen from D, whereas C moves with constant angular velocity ω_t as seen from the equant centre E. Therefore the position of C at the time t can be found in the following way: Having computed $\lambda_m(t)$ from (9.2) by means of the tables of mean motion in longitude, and then $c_m(t)$ from (9.14), we draw through the equant centre a line EC forming an angle $c_m(t)$ with the apogee line. Then EC intersects the deferent at the epicycle centre C.

Mediaeval astronomers found it convenient to draw a line through T parallel to EC and intersecting the ecliptic at M*. It is clear that the angle between T♈ and TM* is equal to the mean longitude λ_m. They called the point M* the mean *motus* of the planet, and TM* the mean *motus* line. This enables us to find the position of C without computing c_m: When λ_m is determined TM* can be drawn. A line parallel to it through E intersects the deferent at C.

It is worth noticing that the angle between TC and either of the parallel lines TM and EC is equal to the equation of centre $q(c_m)$ given by (9.23), while the angle between TC and TP is the prosthaphairesis angle which will be dealt with below.

Finally we have the epicycle vector $\overrightarrow{CP}$ with the constant length $r = 6^p;30$. It rotates with the constant angular velocity ω_a relative to the mean apogee A_m of the epicycle (see page 284) forming with CA_m the mean argument, i.e. the angle $a_m(t)$ given by (9.13). The coupling (9.9) ensures that this rotation produces synodic phenomena (e.g. oppositions when the planet is at the perigee of the epicycle). This explains (9.2b).

The Equation of Argument

The angle $p = PTC$ under which the epicycle radius CP is seen from the Earth is called the *prosthaphairesis of anomaly*, but we shall here prefer the term equation of argument derived from the Latin *aequatio argumenti*. It appears from Figure 9.7 that it can also be defined as the difference

$$p = \lambda - \lambda_c \tag{9.38}$$

15) This is an assumption in the theories of Saturn, Jupiter, and Mars. In the case of Mercury Ptolemy had reasons to believe that the fixed position of the apogee relative to the fixed stars was an observational fact (see below page 312). This may have been a reason for dealing first with the inferior planets, before going on to the superior ones.

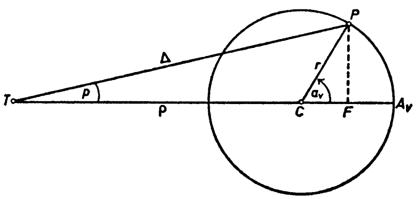


Fig. 9.8

where λ_c is the longitude of the epicycle centre given by (9.25). The value of p as a function of the mean centrum c_m and the true argument a_v can be found from Figure 9.8. Here we first calculate the distance $\Delta = TP$ of the planet from the Earth. We have from the right-angled triangle TFP

$$\Delta^2 = (TC + CF)^2 + FP^2$$

where TC is given by (9.22) and

$$CF = r \cos a_v$$
$$FP = r \sin a_v$$

We have accordingly

$$\Delta(c_m, a_v) = \{(\rho(c_m) + r \cos a_v)^2 + (r \sin a_v)^2\}^{\frac{1}{2}} \tag{9.39}$$

corresponding to (6.51) in the lunar theory.

Now the equation of argument can be expressed as a function of the two variables a_v and c_m by

$$\sin p(a_v, c_m) = \frac{r \sin a_v}{\Delta(c_m, a_v)} \tag{9.40}$$

From (9.38) we have

$$\lambda = \lambda_c + p(a_v, c_m) \tag{9.41}$$

which shows that p is positive when the vector $\overrightarrow{TP}$ (defining λ) precedes the vector $\overrightarrow{TC}$ (defining λ_c), that is for $0° < a_v < 180°$, while it is negative for $180° < a_v < 360°$. This agrees with the sign of sin p in (9.40).

'Testing' the Model

In the Almagest the Saturn theory has been developed up to its present stage from four observations covering a period of $10\frac{1}{2}$ years, which is only about 35 per cent of the complete period of revolution. To build a model on such slender foundations would seem a risky affair. It is, of course, possible that Ptolemy had more observations at his disposal, and only selected those four observations which were most suitable

to demonstrate the parameters already established from a larger variety of data. However, it is understandable that Ptolemy wants to subject his model to what is meant to look like a critical test. As in the lunar theory (page 180) this test is arranged as a control of the mean motions by means of an observation of much greater age than his own. As such he chooses the Babylonian observation P_5, which showed that at the date t_5 = Nabonassar 519, Tybi 14, evening (229 B.C., March 1, 6^h) Saturn had the same longitude as γ Virginis [XI, 7; Hei 2, 419]. Correcting for precession by means of (8.12), this gives the true longitude

$$\lambda(t_5) = 159°;30$$

The corresponding value of the longitude of the apogee is found from (9.37) to be

$$\lambda_a(t_5) = 229°;20$$

Subtracting we find according to (9.14 a) the true centrum

$$c(t_5) = 290°;10$$

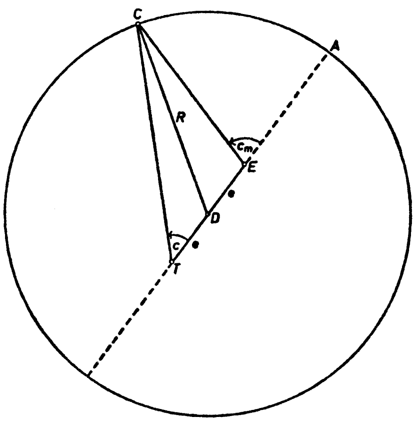


Fig. 9.9

Now the problem is to find the corresponding mean centrum $c_m(t_5)$, and also the mean argument $a_m(t_5)$. The point is that this must be done without using the tables of mean motions since the latter are founded on the very angular velocities ω_t and ω_a which Ptolemy wants to derive. Therefore he has to use the model directly. Figure 9.9 shows that it is possible to calculate the angle c_m when $c = 290°;10$, $R = 60^p$, and $TD = DE = 3^p;25$ are known. Omitting the actual calculation (cf. page 68) we give only the result

$$c_m(t_5) = \lambda_m(t_5) - \lambda_a(t_5) = 283°;33$$

Next we make use of the solar theory to calculate the mean longitude $\lambda_{m\odot}(t_5)$ of the Sun at the time t_5, and accordingly also

$$\lambda_{m\odot}(t_5) - \lambda_a(t_5) = 106°;50$$

Subtracting the two latter equations, and using (9.9), we obtain the empirical value

$$a_m(t_5) = \lambda_{m\odot}(t_5) - \lambda_m(t_5) = 183°;17$$

By means of the quantities determined above we can find empirical values of the mean motions of Saturn in longitude from P_5 to P_3. The corresponding 'theoretical' values can be found from (9.12) and (9.13), that is by entering the tables of mean motions with the time difference $t_3 - t_5 = 364^a\, 219^d\, 18^h$. The results of the two computations are the following

	Empirical	'Theoretical'
$c_m(t_3) - c_m(t_5)$	132°;57	136°;38,46
$a_m(t_3) - a_m(t_5)$	351°;27	351°;26,57

Here the agreement in mean motion in anomaly (mean argument) is perfect. Ptolemy concludes that there is no further reason to make any correction of ω_a as given by (9.5). As to the mean motion in longitude, it looks not quite as good. But here we must remember that the 'theoretical' value 136°;38,46 should be corrected for precession, which in 364 years amounts to about 3°;40 according to the Ptolemaic rate of 1° per century. Subtracting this we get 132°;58,46 which compares fairly well with the empirical value. Thus also ω_t is allowed to remain unchanged from (9.7). The fact that the precession was able to do away with the discrepancy may well have been the main reason why Ptolemy did not admit any proper motion of the planetary apogees (cf. page 287).

So far, so good. But it is obvious that this 'test' comes to nothing if we were right in assuming that the fundamental relation (9.4) is no direct, empirical fact, but derived from observations separated by a long period of time – perhaps from the very observations P_3 and P_5 used for the test itself. In that case it is no wonder that the model can reproduce an observation belonging to its own empirical foundation, and the 'test' becomes nothing more than a pedagogical trick. On the other hand, it might be said that the check on the mean motion in anomaly is a proof that the epicycle radius to the planet has remained parallel to the direction from the Earth to the Mean Sun for a period of almost 400 years; but this is not Ptolemy's conclusion.

The Radices of the Saturn Theory

Having made certain that there is no need to correct either ω_t or ω_a, Ptolemy feels satisfied that the Saturn theory will be able to account for the position of the planet at any given time. Therefore, he does not hesitate to calculate the radices of the motions of Saturn, that is, its mean longitude, mean argument, and apogee at the standard epoch t_0 (Nabonassar 1, Thoth 1, noon). The results are [XI, 8; Hei 2, 425]

$$\begin{aligned} \lambda_m(t_0) &= 296°;43 \\ a_m(t_0) &= \ \ 34°;2 \\ \lambda_a(t_0) &= 224°;10 \end{aligned} \qquad (9.42a–c)$$

as calculated by (9.12), (9.13), and (9.37) respectively.

The Final Formula of the Saturn Theory

The Saturn theory is now complete as far as the motion in longitude is concerned, and we are able to summarize it in a single formula. If we define the position of the planet by its true ecliptic longitude relative to the vernal equinox we find from (9.41) and (9.25)

$$\lambda(t) = \lambda_m(t) + q(c_m) + p(a_v, c_m) \qquad (9.43)$$

where the three terms on the right hand side are determined by (9.12), (9.23), and (9.40) respectively. This means that the calculation of a longitude at the time t must be carried out according to the following procedure:

1) The time interval $(t-t_0)$ is found
2) $\omega_t(t-t_0)$ is taken from the tables
3) $\omega_a(t-t_0)$ is taken from the tables
4) The mean longitude $\lambda_m(t)$ is found from (9.12)
5) The mean argument $a_m(t)$ is found from (9.13)
6) The mean centrum $c_m(t)$ is found from (9.14b)
7) The function $\rho(c_m)$ is found from (9.22)
8) The equation of centre $q(c_m)$ is found from (9.23)
9) The true argument $a_v(t)$ is found from (9.34)
10) The equation of argument $p(a_v, c_m)$ is found from (9.40)
11) The longitude $\lambda(t)$ is found from (9.43)

It should be noticed that in step N° 6 the determination of $c_m(t)$ presupposes that we know $\lambda_a(t)$. Thus if the time t of the calculation is far removed from the standard epoch t_0 we must take precession into account by means of (9.37), which implies one more step in the procedure.

In (9.43) we can subtract the longitude of the apogee λ_a from both sides. According to (9.14a–b) we have the true centrum c(t) in the form

$$c(t) = c_m(t) + q(c_m) + p(a_v, c_m) \qquad (9.44)$$

which gives the position of the planet relative to the apogee. This corresponds more to Ptolemy's usage, but the procedure outlined above remains essentially the same.

It is obvious that it is a cumbersome affair to find the position of the planet in this way. Particularly the steps N^os 7, 8 and 11 involve complicated trigonometrical calculations, which Ptolemy now shows how to avoid by means of a general table of equations, in very much the same way as in the lunar theory (page 196).

The General Table of Equations

In this table [XI, 11; Hei 2, 436] Ptolemy has tabulated the equation of centre (9.23) and the equation of argument (9.40) in such a way that they can be determined

without trigonometrical calculations, and by simple arithmetical operations with the tabulated values. In the preceding chapter [XI, 10; Hei 2, 427] the construction of this table is carefully explained.

Columns I and II contain the arguments of the functions tabulated in the following columns, with intervals of 6° from 0° to 90°, and of 3° from 90° to 180° (cf. the table of lunar equations page 197).

Columns III and IV give the equations of centre, but in a very peculiar way. One could have expected Ptolemy to tabulate (9.23) as he did in the lunar theory (page 197). This would have needed only one column of the table. But instead he tabulates in Column III the function $\alpha(c_m)$ given by (9.26), that is an equation of centre presupposing that the epicycle centre moves upon the equant. In Column IV he then gives the correction $\delta(c_m)$ calculated by the method described on page 281. To find the proper equation of centre we have to compute

$$q(c_m) = \alpha(c_m) + \delta(c_m) \tag{9.45}$$

according to (9.27). This curious method has a plausible explanation if we suppose that Ptolemy had an original *Table of the Equation of Centre* belonging to his First Model of the planetary theories in which the equant was the locus of the epicycle centre. Then he has simply reproduced this table in Column III, with the necessary correction caused by the introduction of the deferent tabulated in Column IV.

It is obvious that it would have been more convenient to combine III and IV into a single column tabulating the equation of centre $q(c_m)$ of the Second Model. This was usually done in Mediaeval versions of the Table.

The last four Columns V–VIII of the Table are concerned with the equation of argument $p(a_v, c_m)$. This is a function of two variables. It is seen from (9.40) and (9.39) that the variable a_v is 'strong' while c_m is 'weak', in the sense defined in Chapter 3 (page 86). Therefore $p(a_v, c_m)$ can be dealt with according to Ptolemy's method of interpolation, almost as in the lunar theory but with one significant difference: In the theory of the Moon p was tabulated as a function of a_v for two standard values of the weak variable c_m, while there are three standard values of c_m in the theory of the superior planets. This makes the standard interpolation method for $p(a_v, c_m)$ a little more complicated.

The Approximation Method for $p(a_v, c_m)$

Let us first consider Column VI containing the tabulated values of a function

$$p_0(a_v) = p(a_v, c_m^0) \tag{9.46}$$

where c_m^0 is a particular value of the mean centrum determined by the condition

$$\rho(c_m^0) = 60^p \tag{9.47}$$

This means that $p_0(a_v)$ is the equation of argument as a function of the 'strong'

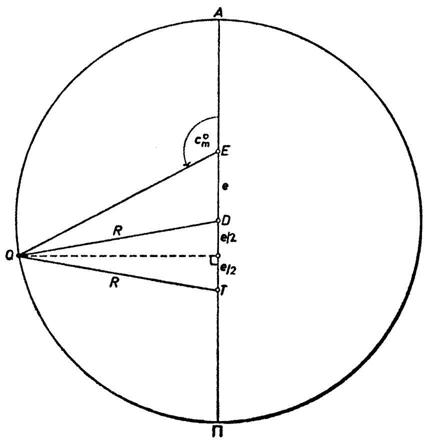


Fig. 9.10

variable a_v only, taken at the point Q of the deferent where the epicycle centre has its mean distance $R = 60^p$ from the Earth. Simple geometrical considerations based on Figure 9.10 show that $c^0{}_m$ is determined by

$$\tan(180^\circ - c_m{}^0) = \frac{2R}{3e}\sqrt{1 - \frac{e^2}{4R^2}} \tag{9.48}$$

For the superior planets this gives the following values. They are not given explicitly in the Almagest, but agree fairly well with what we can infer from the tables.

	$c_m{}^\circ$
Saturn	94°;53
Jupiter	93°;56
Mars	98°;33

If the epicycle centre now moves from Q to the perigee Π of the deferent, the distance $TC = \rho(c_m)$ will decrease from the value $\rho(c_m{}^\circ) = R$ to the value $\rho(180^0) = R - e$ (cf. 9.22). Consequently, for a constant value of a_v, the equation of argument will increase from $p_0(a_v)$ to the greater value

$$p_2(a_v) = p(a_v, 180^\circ) \tag{9.49}$$

The difference

$$p_2(a_v) - p_0(a_v) \tag{9.50}$$

is tabulated in Column VII as a function of a_v.

On the other hand, if C is at the apogee A it has, considered as a function of a_v, the smaller value

$$p_1(a_v) = p(a_v, 0°) \tag{9.51}$$

If C moves towards Q the equation of argument will increase towards $p_0(a_v)$. The difference

$$p_0(a_v) - p_1(a_v) \tag{9.52}$$

is tabulated in Column V. In agreement with (3.57) Ptolemy now postulates that the equation of argument at any position of the epicycle centre can be found from one of the two following formulae [XI, 12; Hei 2, 446]:

If the mean centrum lies in the interval $0° < c_m < c_m{}^0$ we have

$$p(a_v, c_m) = p_0(a_v) - \{p_0 - p_1\} \frac{f'(c_m)}{60} \tag{9.53}$$

but if c_m lies in the interval $c_m{}^0 < c_m < 180°$ we have

$$p(a_v, c_m) = p_3(a_v) + \{p_2 - p_0\} \frac{f''(c_m)}{60} \tag{9.54}$$

Here $f'(c_m)$ and $f''(c_m)$ are two functions of the mean centrum tabulated in Column VIII. When c_m goes from 0° to $c_m{}^0$, $f'(c_m)$ decreases from 60 to zero, and when c_m goes from c_m to 180° $f''(c_m)$ increases from zero to 60. The two functions are determined by the maximum values of the equation of argument in agreement with (3.58) so that we here have [XI, 10; Hei 2, 434]

$$f'(c_m) = 60 \cdot \frac{\max p(c_m) - \max p(0°)}{\max p(c_m{}^0) - \max p(0°)} \tag{9.55}$$

and

$$f''(c_m) = 60 \cdot \frac{\max p(\ c_m\ ) - \max p(c_m{}^0)}{\max p(180°) - \max p(c_m{}^0)} \tag{9.56}$$

CHAPTER 10

The Inferior Planets

Introduction

Having dealt with the theories of the Sun, the Moon, and the superior planets, we are left with the problem of describing the motions of Venus and Mercury. The reason why these planets are called inferior is that according to the usual order of the celestial bodies (see page 261) they are situated below the Sun, or, more precisely, between the orbits of the Sun and the Moon. Now in ancient astronomy the order of the planets is more or less a convention, without any sure foundation upon observable facts. Thus Ptolemy reminds us that there is no perceptible parallax in any of the planets, and that one has never observed a passage of Venus or Mercury before the disc of the Sun. He even maintains that such a passage has never occurred; but on the other hand this latter circumstance is no proof that there are no planets below the Sun since their orbits could be so inclined that a conjunction never results in a passage [IX, 1; Hei 2, 207]. It is clear, however, that it is too superficial to distinguish the inferior from the superior planets by their conventional and essentially arbitrary order.

Nevertheless, Venus and Mercury exhibit one particular phenomenon clearly distinguishing them from their superior counterparts: they are never in opposition, and therefore never seen around midnight. On the contrary, they are always seen in the neighbourhood of the Sun, the *maximum elongation* being 47° for Venus and between 18° and 28° for Mercury. When the elongation is eastern (and great enough) these planets are, therefore, seen as evening stars above the western horizon just after sunset. When it is western they are seen as morning stars above the eastern horizon just before sunrise. In between there is a *period of invisibility* during which the planets are too near to the Sun to be seen. At a certain time during this period they will be in an invisible or *lower conjunction* with the Sun, no upper conjunction being possible. It follows that the most important synodic phenomena will be the maximum eastern and western elongations. In fact the theories of both Venus and Mercury are constructed mainly from observations of such maximal elongations, in contradistinction to the oppositions used as the fundament of the theories of the superior planets.

Also in the inferior planets we can distinguish two different 'anomalies', or deviations from uniform circular motion. Since roughly speaking they follow the Sun around the ecliptic, it is clear that they must exhibit a *first anomaly* related to their position upon the ecliptic. This is similar to the motion of the superior planets, and we can also here define a mean tropical angular velocity ω_t, called the mean motion in longitude, and related to a mean tropical period T_t by the relation $\omega_t \cdot T_t = 360°$.

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_10,

Since both planets accompany the Sun on its annual course it follows that we must have

$$T_t = \text{one tropical year, and } \omega_t = \omega_\odot \tag{10.1}$$

The second anomaly is related to the position of the planets relative to the Sun and responsible for the synodic phenomena, exactly as in the case of the superior planets. This means that the relations (9.2) are valid for Venus and Mercury so that here too it is unnecessary to distinguish between a synodic and an anomalistic period[1]). In the following we shall consider the anomalistic period T_a and the corresponding mean motion in anomaly ω_a defined by $\omega_a \cdot T_a = 360°$ as the fundemental kinematic parameter of the theory.

The Mean Motions of Venus and Mercury

As in the cases of the superior planets, the daily mean motions are calculated by means of relations of a kind similar to those used for Saturn, Jupiter, and Mars (cf. page 270). Ptolemy quotes [IX, 3; Hei 2, 215] the following:

Venus: 5 returns in anomaly take place together with 8 revolutions − 2°;15, in $8^a - 2^d;18$

Mercury: 145 returns in anomaly take place together with 46 revolutions + 1°, in $46^a + 1^d;2$

This gives for Venus the daily mean motion in longitude

$$\omega_t = \frac{8 \cdot 360° - 2°;15}{8 \cdot 365^d;14,48 - 2^d;18} = 0°;59,8,17,13,12,31$$

$$= \omega_\odot \text{ (cf. 5.11)} \tag{10.1a}$$

and in anomaly

$$\omega_a = \frac{5 \cdot 360°}{8 \cdot 365^d;14,48 - 2^d;18}$$

which, with a small correction explained later, gives

$$\omega_a = 0°;36,59,25,53,11,28 \tag{10.2}$$

1) For Venus and Mercury text books of astronomy often derive the relation

$$\frac{1}{T} = \frac{1}{T_t} - \frac{1}{T_a}$$

as the analogy to (9.11). But then the period T_t is not one tropical year, but the tropical period of one of the inner planets as defined in a heliocentric system. This concept is foreign to Ptolemaic astronomy to which the relation above is without interest. Cf. Dreyer (1906, pp. 154 and 197).

For Mercury we find in a similar way

$$\omega_t = 0°;59,8,17,13,12,31 \tag{10.3}$$

$$\omega_a = 3°;6,24,6,59,35,50 \tag{10.4}$$

The fact that $\omega_t = \omega_{\odot}$ has immediate consequences for the tables of these planets, since the mean motion in longitude can be tabulated simply by copying the tables of the mean motus of the Sun, as Ptolemy actually does [IX, 4; Hei 2, 244]. Therefore these tables express nothing more than the linear function (5.2) known from the solar theory. On the other hand the tables of mean motion in anomaly correspond to a function similar to, for instance, (6.7) but with the proper value of ω_a for each planet.

The Choice of a Geometrical Model

The fact that both Venus and Mercury have a daily mean motion in longitude exactly equal to that of the Sun can be accounted for in at least two different ways known to astronomers in late Antiquity[2]).

Thus the 4th century B.C. the astronomer Heracleides of Pontus has been credited with a model in which both Venus and Mercury rotate around the Sun at the same time as the latter rotates about the Earth (the geo-heliocentric system). In other words, the orbits of Venus and Mercury are supposed to be epicycles, both having their centres in the Sun and travelling along the ecliptic together with it. This hypothesis would explain that each planet has a limited elongation, and that its apparent place on the ecliptic performs a kind of oscillatory motion around the Sun, superimposed on the annual eastward movement of the latter. It follows that the mean positions of the planets are identical and equal to the true position of the Sun.

Even if the geo-heliocentric system was always remembered by later commentators, it was regarded more as a historical curiosity than as a serious astronomical hypothesis. Ptolemy does not mention it, but sticks to his general idea that any planet moves upon an epicycle, which again has its centre moving upon a deferent circle of its own. Models of this kind work well for the superior planets, where the motion in longitude was connected with the deferent and took place with an individual angular velocity, whereas the motion in anomaly was described by means of the epicycle and coupled to the motion of the Sun.

Now it is immediately seen that if a similar model is going to work for the inferior planets, then it will be necessary to change the roles of deferent and epicycle. Here it is the daily motion in longitude ω_t which is equal to that of the Sun. Therefore the coupling between the planet and the Sun necessary to produce the synodic phenomena must be established by means of the deferent, while the motion on the epicycle can be regarded as independent and characteristic of the individual planet. This Ptolemy

2) For earlier conceptions of the motions of the inferior planets, see Saltzer (1970). – Neugebauer (1972) argues that the ascription of the geo-heliocentric system to Heracleides rests upon a mistake by Chalcidius in his commentary on Plato's *Timaios*.

achieves simply by giving the epicycle centre C – marking the mean position of the planet – the same ecliptic longitude $\lambda_{m\odot}$ as that of the mean Sun (cf. page 133). Thus the epicycle centre revolves upon an eccentric deferent, completing a revolution in one tropical year, while the planet itself revolves upon the epicycle with its own characteristic period. Looking at the theories from a Copernican point of view (cf. page 285) we see that in the case of the inferior planets the annual motion of the Earth is reflected in the motion upon the deferent, while in the case of the superior planets it was reflected in the motion upon the epicycle.

We shall now see how Ptolemy derives the parameters of both the Venus and the Mercury model from observations[3]). Although he begins with Mercury we shall first consider Venus, the theory of which is more like those of the superior planets. In this chapter we are concerned only with longitudes, the theories of latitude being dealt with later (page 368). This implies that all the lines and circles of the models can be regarded as situated in the plane of the ecliptic.

The empirical data of the Venus theory

The observations upon which the theory of Venus are founded are collected in the following table. Here we find in

Column I a symbol V_i denoting the particular observation number i;
Column II gives the time t_i of V_i as it is found in the Almagest;
Column III gives the date of the observation in the Julian calendar, according to Manitius;
Column IV contains the observed true longitude $\lambda(t_i)$ of Venus at the time t_i.
Column V lists the corresponding longitude $\lambda_{m\odot}(t_i)$ of the Mean Sun, calculated from the solar theory;
Column VI gives the numerical values of the maximum elongation of Venus from the Mean Sun, and
Column VII indicates the number of V_i in the General Table of Observations (Appendix A).

All of the 11 observations were made at Alexandria, V_1 and V_2 by Timocharis, V_3–V_5 by Theon of Smyrna who gave them to Ptolemy [X, 1; Hei 2, 296], and the rest by Ptolemy himself[4]).

3) For a general account of the Ptolemaic theories of the inferior planets see Delambre (II, pp. 308–346), Herz (1887, pp. 109 ff.), and Boelk (1911). The most lucid explanation of the theory of Mercury is due to W. Hartner (1955).
4) A critical analysis of the observations of Venus quoted in the *Almagest* has been given by Czwalina (1959) who calculated the times of maximal elongations and compared them with the Ptolemaic data. His conclusion (p. 17) is that the discrepancies between observed and theoretical values can be explained on the assumption that Ptolemy was unable to measure maximal elongations to an accuracy better than ± 10.6 minutes of arc. – See also Wilson (1972).

I	II	III	IV	V	VI	VII
V_1	Phil. 13 (Nab. 476) Mesore 17/18 12^h of the night	B.C. 272 Oct 12	154°;10			18
V_2	Phil. 13 (Nab. 476) Mesore 21/22 Morning	B.C. 272 Oct 16	158°;50			19
V_3	Hadr. 12 (Nab. 874) Athyr 21/22 Morning	A.D. 127 Oct 11	150°;20	197°;52	47°;32	56
V_4	Hadr. 13 (Nab. 875) Epiphi 2/3 Morning	A.D. 129 May 20	10°;36	55°;24	44°;48	57
V_5	Hadr. 16 (Nab. 878) Pharmuti 21/22 Evening	A.D. 132 March 7	31°;30	344°;15	47°;15	61
V_6	Hadr. 18 (Nab. 880) Pharmuti 2/3 Morning	A.D. 134 Feb 18	281°;55	325°;30	43°;35	66
V_7	Hadr. 21 (Nab. 883) Tybi 2/3 Evening	A.D. 136 Nov 18	282°;50	235°;30	47°;20	76
V_8	Hadr. 21 (Nab. 883) Mechir 9/10 Evening	A.D. 136 Dec 25	319°;36	272°; 4	47°;32	77
V_9	Ant. 2 (Nab. 885) Tybi 29/30 $4\frac{3}{4}^h$ after midnight	A.D. 138 Dec 16	216°;30	262°; 9		80
V_{10}	Ant. 3 (Nab. 886) Pharmuti 4/5 Evening	A.D. 140 Feb 18	13°;50	325°;30	48°;20	90
V_{11}	Ant. 4 (Nab. 887)[5] Thoth 11/12 Morning	A.D. 140 July 30	78°;30	125°;45	47°;15	93

5) In Heiberg's edition of the Almagest [X, 1; Hei 2, 297] the date of the observation V_{11} is given as the 14th year of Antoninus. This year is also found in Delambre (II, p. 333) who only wondered that observations with the naked eye were able to give exactly equal maximal eastern elongations (cf. V_5 and V_{11}). However, in his German translation (Vol. 2, p. 412, note 11) Manitius found it difficult to believe that Ptolemy used an observation at so late a date. He proposed accordingly to place V_{11} in the 4th year of Antoninus, considering the Greek ιδ′ (= 14) to be a scribal error for δ′ (= 4). This emendation was confirmed by direct calculation by Czwalina (1959, p. 17).

The Apogee of Venus

In the cases of the Sun and the superior planets the position of the apsidal line – characterized by the ecliptic longitude λ_a of the apogee – was determined in the course of the same calculation as that which led to the eccentricity. In the Venus theory the apogee can be found independently by a method based upon observations of maximum eastern or western elongations, and implying a high degree of symmetry, due to which the construction of the theory is simpler than in the former cases. The method rests on the fact that the apparent radius of the epicycle – measured simply by the maximum elongation – is a function of the position of the epicycle on the eccentric circle, and that this function is symmetrical with respect to the apsidal line. Therefore Ptolemy selects [X, 1; Hei 2, 296] pairs of observations with equal but opposite maximum elongations. Such a pair is V_5 and V_{11}, and another is V_3 and V_8. The following Figure 10.1 shows the positions of the epicycle corresponding to these four observations.

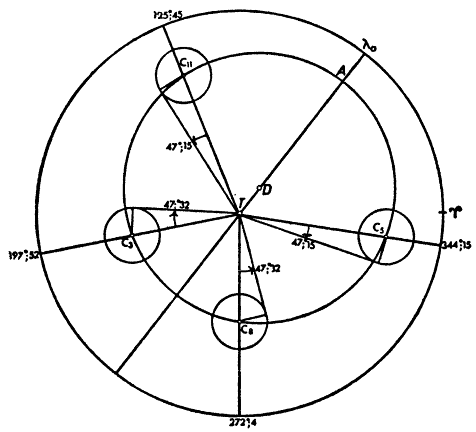


Fig. 10.1

It is obvious that the apsidal line must bisect the angle between the centres C_5 and C_{11} as seen from the Earth T. Since the ecliptic longitudes of these centres are equal to the longitudes of the Mean Sun at the times t_5 and t_{11}, the apsidal line must pass through a point of the ecliptic with the longitude

$$\frac{\lambda_{m\odot}(t_5) + \lambda_{m\odot}(t_{11})}{2} = \frac{344°;15 + 125°;45}{2} = 55°;0$$

as well as through the opposite point with the longitude 235°. In order to know which of these points is the apogee we only have to consider V_3 and V_8. This pair of observations are symmetrical about a line through T and a point with the longitude

$$\frac{\lambda_{m\odot}(t_3) + \lambda_{m\odot}(t_8)}{2} = \frac{197°;52 + 272°;4}{2} = 54°;58$$

This is practically the same as before. But since the epicycle looks larger at V_3 and V_8 (where the maximum elongation is 47°;32 according to the table page 299) than at V_5 and V_{11} (where the maximum elongation is 47°;15) it is clear that it must be nearer to the Earth at the former position than at the latter. Ptolemy concluded therefore that the apogee of Venus has the ecliptic longitude

$$\lambda_a = 55° \tag{10.5}$$

and the perigee a longitude of 235°.

As in the cases of the superior planets, the apogee is supposed to be a fixed point relative to the fixed stars, Consequently, it takes part in the slow movement of precession relative to the vernal equinox. Ptolemy tries to make this supposition plausible by determining the sum of the maximum eastern and western elongation at various times [X, 2; Hei 2, 300]. We see from the table above that at the time t_7 the epicycle centre is passing the perigee, and that the sum of the elongations is $2 \cdot 47°;20 = 94°;40$. At the time t_4, seven years before, it was at the apogee and the sum was $2 \cdot 44°;48 = 89°;36$. Ptolemy further maintains that these values are the absolute maximum and minimum of the sum respectively, since at any other place of the ecliptic, i.e. at any other time of observation, the corresponding sum lies in between these two numbers. This would not be the case if the apogee were moving, and we might conclude that it has a fixed longitude. It appears later [X, 4; Hei 2, 311] that this is true for short intervals of time only. During long intervals the apogee is supposed to move 1° per 100^a like the fixed stars.

The Eccentricity and the Size of the Epicycle

We have seen above (page 276) how in the case of the superior planets Ptolemy first determined the circle of uniform motion, or the equant, before, as a second approximation, he determined the eccentric deferent upon which the epicycle centre really moves by halving the eccentricity of the equant. In the case of the inferior planets he proceeds in the opposite way. The circle in Figure 10.1 is the real deferent carrying the epicycle centre around its centre D. The distance $TD = e$ is the eccentricity, which is easily found together with the radius r of the epicycle. To that purpose Ptolemy first selects the observation V_4, for which the longitude of the epicycle centre C_4 is $\lambda_{m\odot}(t_4) = 55°;24$, or very nearly equal to the longitude $\lambda_a = 55°$ of the apogee. At V_4 the apparent radius of the epicycle is 44°;48. Next he considers V_7, at which the centre C_7 has a longitude of 235°;30 and lies very nearly at the perigee. Here the apparent radius of the epicycle is 47°;20. Figure 10.2 shows these two situations. Here

$$C_4T = C_4D + DT = R + e$$
$$C_7T = C_7D - DT = R - e$$

From the two triangles we then have

$$r = (R + e) \cdot \sin 44°;48$$
$$r = (R - e) \cdot \sin 47°;20$$

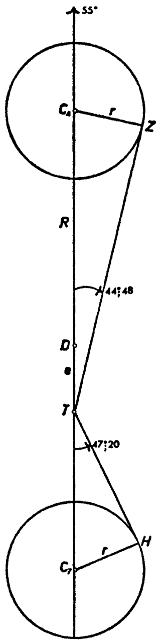


Fig. 10.2

These two equations contain three unknown quantities e, r, and R, one of which has to be given an arbitrary value. As in the preceding models we choose the standard value $R = 60^p$ of the deferent radius. Eliminating r we obtain

$$\frac{e}{R} = \frac{\sin 47°;20 - \sin 44°;48}{\sin 47°;20 + \sin 44°;48} \qquad (10.6)$$

and eliminating e

$$\frac{r}{R} = \frac{2 \cdot \sin 44°;48 \cdot \sin 47°;20}{\sin 44°;48 + \sin 47°;20} \qquad (10.7)$$

With $R = 60^p$ these expressions give the eccentricity

$$e = 1^p;15 \qquad (10.8)$$

and the radius of the epicycle

$$r = 43^p;10 \qquad (10.9)$$

The Complete Model

We can now describe the complete model of the Venus theory by the relation

$$\overrightarrow{TP} = \overrightarrow{TD} + \overrightarrow{DC} + \overrightarrow{CP} \qquad (10.10)$$

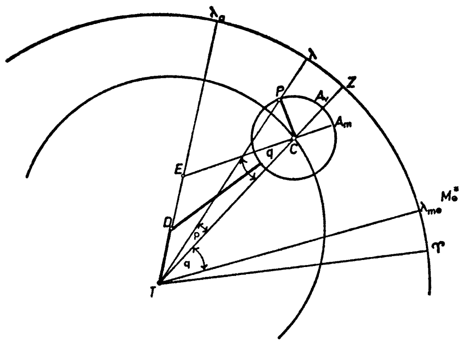


Fig. 10.3

which is understood from Figure 10.3. Here the geocentric position vector $\overrightarrow{TP}$ is resolved into three components.

1) The eccentricity vector $\overrightarrow{TD}$ has the constant length $e = 1^p;15$ and a fixed direction relative to the fixed stars. With the direction to the vernal equinox it forms an angle which can be expressed by the general formula (9.37).

2) The deferent vector $\overrightarrow{DC}$ has the constant length $R = 60^p$. It rotates about D with a varying angular velocity and a period of 1 tropical year. However, the motion of the epicycle centre C is supposed to be uniform with respect to an equant point E on the apsidal line, the position of which will be determined below (page 304).

3) The epicycle vector $\overrightarrow{CP}$ has the constant length $r = 43^p;10$ and rotates about C with the constant angular velocity ω_a relative to the mean apogee A_m of the epicycle. Here, as with the superior planets, A_m is defined as the point on the epicycle furthest away from E, whereas the true apogee A_v of the epicycle is the point furthest away from the Earth.

Thus the theory of Venus is very similar to those of the superior planets (cf. page 286) and we can introduce a similar set of variables to describe its motion. The mean and true longitude $\lambda_m(t)$ and $\lambda(t)$, the mean and true centrum $c_m(t)$ and $c(t)$, and the mean and true argument $a_m(t)$ and $a_v(t)$ are defined in the same way as the corresponding variables in Chapter 9. Also the equation of centre $q(c_m)$ and the equation of argument $p(a_v, c_m)$ are still determined by (9.23) and (9.40) respectively – but only if the centre E of the uniform motion has the distance $TC = 2e$ from the centre of the Earth; that this is so is proved in the following section. It follows that the fundamental relations between the variables (9.25), (9.34), and (9.41) are still valid, so that the complete theory of Venus' motion in longitude can be summarized by the final formula (9.43), resp. (9.44). Actually the only difference between the theory of Venus and those of the superior planets is that in the Venus model it is the line EC which is parallel to the mean *motus* line $TM_{\odot}{}^{*}$ of the Sun, while it is the line CP in the superior planets. This implies that (9.9) must be replaced by the relation

$$\lambda_m(t) = \lambda_{m\odot}(t) \tag{10.11}$$

in accordance with what was stated above (page 298, cf. 10.1). Therefore, (9.43) can be written as

$$\lambda(t) = \lambda_{m\odot}(t) + q(c_m) + p(a_v, c_m) \tag{10.12}$$

while (9.44) remains unchanged.

In the case of the inferior planets a new relation for the equation of centre $q(c_m)$ can be deduced in the following way. Let us consider a definite position of the epicycle centre C, characterized by the mean centrum c_m, or the corresponding mean longitude $\lambda_{m\odot}$. Let the maximum eastern elongation occur when the true argument has the value $a_v{}^e$, and the maximum western elongation when it has the value $a_v{}^w$. According to (10.12) the two extreme longitudes are given by

$$\lambda^e = \lambda_{m\odot} + q(c_m) + p(a_v{}^e, c_m) \tag{10.13 a}$$

$$\lambda^w = \lambda_{m\odot} + q(c_m) + p(a_v{}^w, c_m) \tag{10.13 b}$$

Because the two extreme elongations correspond to symmetrical positions of Venus on the epicycle, we have

$$p(a_v{}^e, c_m) = -p(a_v{}^w, c_m) \tag{10.14}$$

Therefore adding the two equations above we obtain

$$\lambda^e + \lambda^w = \lambda_{m\odot} + \lambda_{m\odot} + 2q(c_m)$$

or

$$q(c_m) = \tfrac{1}{2}\{(\lambda^e - \lambda_{m\odot}) - (\lambda_{m\odot} - \lambda^w)\} \tag{10.15}$$

In other words, the equation of the centre is half of the difference between the maximum eastern and western elongations of the true planet from the Mean Sun. Of course, a similar relation does not exist in the case of the superior planets, which have no maximum elongations.

The Introduction of the Equant

The only remaining problem connected with the model is to investigate *whether the uniform motion of the epicycle takes place around the point D* [X, 3; Hei 2, 302], that is around the centre of the real deferent determined above by the eccentricity $TD = e = 1^p;15$. Ptolemy investigates this problem by assuming that in fact there is a point E around which the motion of C is uniform. To find the position of E he makes use of the observations V_6 and V_{10}, which are selected for two related reasons. First, it makes the calculations easier that in both cases the mean centrum is

$$c_m(t_6) = c_m(t_{10}) = \lambda_{m\odot} - \lambda_a = 270°;30$$

so that EC is very nearly perpendicular to the apogeum line. Second, the fact that $c_m(t_6) = c_m(t_{10})$ makes it possible to use (10.15).

Let us first realize that the possible transition to an equant model does not invalidate the values of λ_a, e, and r found above, if only E is supposed to lie somewhere on the apogeum line. In that case the symmetries used for determining λ_a will still hold good, and the same value of λ_a will come out. Similarly e and r were found from observations at the apogee and perigee, for which the ecliptic longitude C is unaffected by a displacement of the observer from T to E. But it is important to notice that when C is outside the apogeum line it is not TC which points towards the Mean Sun, but EC, or rather the line through T parallel to EC. This was called the *linea medii motus* by Mediaeval Latin astronomers (cf. page 287).

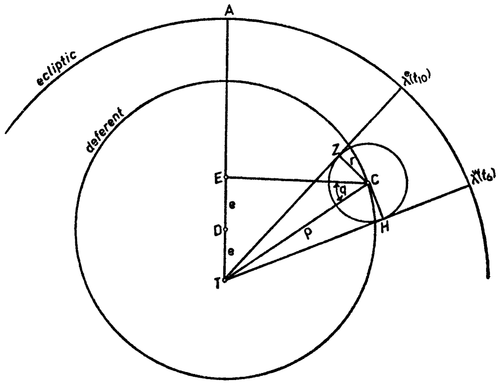


Fig. 10.4

Next, let us construct the position of the epicycle centre, drawing a line through E forming an angle $c_m = 270°;30$ with the apogeum line and intersecting the deferent at C, around which point the epicycle is drawn with the radius $43^p;10$ (see Figure 10.4). Next we draw the two tangents TH and TZ which (prolonged to the ecliptic) mark the points $\lambda^w(t_6)$ and $\lambda^e(t_{10})$ of greatest western and eastern elongation respectively. The tangents include an angle of 91°;55, bisected by TC = $\rho(c_m)$. In triangle TCZ we now have Z = 90°, T = $\frac{1}{2} \cdot 91°;55 = 45°;57,30$, and CZ = r = $43^p;10$. We can then find

$$\rho(c_m) = TC = \frac{43^p;10}{\sin 45°;57,30} = 60^p;3$$

In triangle ECT we have E ≈ 90° and C = $q(c_m)$, which according to (10.15) can be found from the maximum elongations as

$$q(c_m) = \tfrac{1}{2}\{48°;20 - 43°;35\} = 2°;22,30$$

We then have

$$TE = TC \cdot \sin q(c_m) = 60^p;3 \sin 2°;22,30 = 2^p;30 \qquad (10.16)$$

This means that TE is twice TD, or that D is the mid point of TE, exactly as in the cases of the superior planets. Since we denoted TD by e, we have TE = 2e, which in the following we shall call the eccentricity of the equant.

The Bisection of the Eccentricity Reconsidered

This way of introducing the equant point E is very different from what Ptolemy did in the cases of the superior planets, where he first determined the equant circle from 3 oppositions, and next the deferent by bisection of the eccentricity of the equant. The reasons for this bisection were not explained, but one hypothesis was that it was the first step in an iterative process (page 279).

In Venus the true deferent came first, and the equant was introduced later. Consequently there is no question here of bisecting the eccentricity of the equant. Rather, one could speak of a doubling of the eccentricity of the deferent, but this would be misleading. For although it is true that TD = e is increased by a factor 2 in order to give the distance TE, this factor is by no means arbitrary. It emerges as the ratio $2^p;30 : 1^p;15$ between two numbers both of which are the outcome of calculations based directly upon empirical data. This contrasts very obviously with the factor 2 by which the eccentricity was decreased with the superior planets.

In view of this state of affairs another hypothesis emerges as a possible explanation of the bisection. It could very well be that Ptolemy had reasoned as follows: In the theory of Venus we are able to determine first the actual deferent with the centre D and the eccentricity e, and next the centre E of the circle of uniform motion with an eccentricity which proves to be 2e, so that $TD = \frac{1}{2} \cdot TE$. On the other hand, in the cases of the superior planets we were able to determine first the circle of uniform motion and its eccentricity; next we had to introduce a deferent circle carrying the epicycle and having a reduced eccentricity; its centre D was unknown, but it would be worth trying if D were not in the middle between T and E, since this was the case for Venus.

This hypothesis would explain the words actually used by Ptolemy, viz. that *we found that the epicycle centres revolve upon circles of the same size as the circles producing the anomaly, but with centres in the middle points* of the lines between E and T [IX, 5; Hei 2, 252].

This second hypothesis has two obvious implications. First, that the words *we found* are ambiguous since in the theory of Venus we found the position of both D and E in a purely empirical way, while for the superior planets we found the position of D by trial and error from the empirical position of E. Next, the hypothesis implies that Ptolemy must have constructed the theory of Venus before he dealt with the superior planets.

Now we do not know in which order Ptolemy worked out the models of the five planets. In the Almagest they seem to be placed relative to each other mainly by a formalistic principle, viz. that the order of exposition ought to follow the order in

which nature is supposed to have placed the planets. But it is possible that Ptolemy actually did his original work in a quite different order, or that he worked on all the planetary theories at roughly the same time. Perhaps there is a hint of a possible solution of this problem if we look at the dates at which Ptolemy finished the collecting of observational material for the individual models. These data can be divided into two classes: 1) observations of oppositions, or maximum elongations, for the determination of equant and deferent circles, and 2) observations of simple longitudes for other purposes. From Appendix A we can collect the following dates.

Latest observations

	Class I Oppositions or maximum elongations	Class II Simple longitudes
Mercury	A.D. 141 Feb 2 (N° 94)	139 May 17 (N° 84)
Venus	140 July 30 (N° 93)	138 Dec 16 (N° 80)
Mars	139 May 27 (N° 85)	139 May 30 (N° 86)
Jupiter	137 Oct 8 (N° 78)	139 July 11 (N° 88)
Saturn	136 July 8 (N° 74)	138 Dec 22 (N° 81)

Before discussing these dates we should remember that it is by no means obvious that Ptolemy finished one of the models as soon as he had made the latest relevant observation quoted in the Almagest. As mentioned before (page 288 f.), it is quite possible that these quoted observations were selected because they were suitable for deducing parameters which had been obtained from a more abundant collection of data. But if the dates listed above reveal anything at all, it is rather that Ptolemy seems to have worked his way backwards, beginning with Saturn, and finishing with Mercury. This is, of course, a far from certain conclusion. Nevertheless, it makes it more likely than not that Ptolemy dealt first with the superior planets, and afterwards with the inferior ones.

If this be so then the bisection of the eccentricity of the superior planets seems more understandable in terms of the iterative procedure described above, than as resulting from an analogy with the theories of the inferior planets. In any case it is difficult to see how a really satisfactory solution can be found.

A 'Test' of the Theory

Before determining the initial values $\lambda_m(t_0)$ and $a_m(t_0)$ of the mean longitude and mean argument, Ptolemy asks: which value should we give the daily mean motion in anomaly ω_a in order to make the model describe the observations [X, 4; Hei 2, 306]? This question is investigated by a comparison of the observations V_9 and V_1, both of

which give a position of Venus different from the maximum elongation. Without doubt these observations are chosen because the times t_1 and t_9 are more sharply defined than the time of a maximum elongation, where the variation in longitude is small during a considerable period of time.

The two observations provide us with the following data (cf. the table page 299)

	V_1	V_9
$t - t_0$	$475^a346^d18^h$	$885^a148^d16^h;45$
$\lambda(t)$	$154°;10$	$216°;30$
$\lambda_{m\odot}(t)$	$197°;3$	$262°;9$

Furthermore, taking precession into account we have the following apogees

$\lambda_a(t)$	$50°;55$	$55°$

From these values the model enables us to compute

$a_m(t)$	$252°;7$	$230°;32$

This is done by means of a direct trigonometrical calculation which may be omitted here. The time difference is $t_9 - t_1 = 409^a167^d$, during which Venus has performed 255 complete revolutions in anomaly (cf. the period relation page 296), plus the extra amount $a_m(t_9) - a_m(t_1) = 338°;25$. Thus the daily mean motion in anomaly is

$$\omega_a = \frac{255 \cdot 360° + 338°;25}{409 \cdot 365^d;14,48 + 167^d} \tag{10.17}$$

which gives the value already quoted (10.2), from which the tables of mean motion in anomaly are computed.

The mean motion in longitude ω_t being equal to that of the Sun (5.1), Ptolemy feels no desire to subject it to any particular test beyond what was performed in the solar theory (page 154).

The Radices of the Venus Theory

We can now determine the initial values of the kinematic parameters of the Venus theory referred to the standard epoch t_0, that is the beginning of the era of Nabonassar [X, 5; Hei 2, 315].

As for the mean motion in longitude, we know already from (5.30) and (10.11) that

$$\lambda_m(t_0) = \lambda_{m\odot}(t_0) = 330°;45 \tag{10.18}$$

To compute the *radix* (initial value) of the mean motion in anomaly Ptolemy uses the observation V_1. A calculation similar to that of the theories of the superior planets gives

$$a_m(t_0) = 71°;7 \tag{10.19}$$

and finally the position of the apogee at the time t_3 is found from (9.37)

$$\lambda_a(t_0) = 46°;10 \tag{10.20}$$

The General Table of Equations

The Venus theory is now complete, and the only remaining problem is how to make it suitable for practical computations. As in the previous models, all trigonometrical calculations are avoided by means of a *General Table of Equations* [XI, 11; Hei 2, 442] which is constructed exactly as in the cases of the superior planets as explained page 291. We shall not repeat this part of the theory here, but only notice that the practical determination of the equation of the centre $q(c_m)$ is carried out by a procedure corresponding to (9.45), and that the equation of argument $p(a_v, c_m)$ is calculated by (9.53) or (9.54).

Thus Ptolemy has achieved a high degree of uniformity, the theory of Venus being cast in very much the same mould as the theory of the superior planets. However, this beautiful scheme breaks down in the following, where we are going to consider the Mercury model which defies the general pattern and derives its theoretical features more from the Moon than from the other planets.

The Mercury Observations

As developed in the Almagest the theory of Mercury is based on a total of 16 observations which are listed in chronological order in the following table. The order and contents of the columns correspond to those of the table of observations of Venus (page 299), except that not all the elongations in Colum VI are maximum elongations. Among the observations Ptolemy is himself responsible for M_9–M_{16}. M_8 is found in the scientific remains of Theon of Smyrna (see page 298), and M_6–M_7 are presumably of Babylonian origin. M_1–M_5 are dated by the era of Dionysios and would seem to have been made by this Hellenistic astronomer, or perhaps by Timocharis who made Venus-observations at about the same time (see page 298). M_4 is quoted according to the calculation of Hipparchus, and must be an older observation reduced by the great astronomer.

Because of the very peculiar nature of the final Mercury theory some attention has been given to these observations, a certain number of which are very erroneous (see page 314). However, a systematic criticism of them would be outside the scope of the present work, which is not concerned with Ptolemy as an observer, but with the general structure of his theoretical astronomy[7]).

7) A critical examination of the observations of Mercury used by Ptolemy was made by Czwalina (1959) who made it plausible that both M_{11} and M_{12} took place in the 18th year of Hadrian (i.e. before M_{10}). His dating of M_3 to Mechir 30 instead of Phamenoth 30 (in disagreement with Heiberg's text [IX, 7: Hei 2, 265]) was already proposed by Ideler and adopted by Manitius (see the German translation, vol. 2, p. 133).

Mercury Observations in the Almagest

I	II	III	IV	V	VI	VII
M_1	Dion. 21 (Nab. 483) Scorpion 22 (Thoth 18/19) Morning	B.C. 265 Nov 15	213°;20	230°;50		20
M_2	Dion. 23 (Nab. 486) Hydron 21 (Choiak 17/18) Morning	B.C. 262 Feb 12	292°;20	318°;10	25°;50	22
M_3	Dion. 23 (Nab. 486) Tauron 4 (Mechir 30) Evening	B.C. 262 April 25	53°;40	29°;30	24°;10	23
M_4	Dion. 24 (Nab. 486) Leonton 28 (Payni 30) Evening	B.C. 262 Aug 23	169°;30	147°;50	21°;40	24
M_5	Dion. 28 (Nab. 491) Didymon 7 (Pharmuti 5/6) Evening	B.C. 257 May 28	89°;20	62°;50	26°;30	25
M_6	Chald. 67 (Nab. 504)[6]) Apellaios 5 (Thoth 27/28) Morning	B.C. 245 Nov 19	212°;20	234°;50	22°;30	26
M_7	Chald. 75 (Nab. 512) Dios 14 (Thoth 9/10) Morning	B.C. 237 Oct 30	194°;10	215°;10	21°	28
M_8	Hadr. 14 (Nab. 876) Mesore 18 Evening	A.D. 130 July 4	126°;20	100°;5	26°;15	58
M_9	Hadr. 16 (Nab. 878) Phamenoth 16/17 Evening	A.D. 132 Feb 2	331°	309°;45	21°;15	60
M_{10}	Hadr. 18 (Nab. 880) Epiphi 18/19 Morning	A.D. 134 June 4	48°;45	70°	21°;15	67
M_{11}	Hadr. 19 (Nab. 881) Athyr 14/15 Morning	A.D. 134 Oct 3	170°;12	189°;15	19°;3	68
M_{12}	Hadr. 19 (Nab. 881) Pachon 19 Evening	A.D. 135 April 5	34°;20	11°;5	23°;15	71
M_{13}	Anton. 1 (Nab. 884) Epiphi 20/21 Evening	A.D. 138 June 4	97°	70°;30	26°;30	79
M_{14}	Anton. 2 (Nab. 885) Epiphi 2/3 Evening	A.D. 139 May 17	77°;30	52°;34		84
M_{15}	Anton. 2 (Nab. 885) Mesore 21 Morning	A.D. 139 July 5	80°;5	100°;20	20°;15	87
M_{16}	Anton. 4 (Nab. 887) Phamenoth 18/19 Morning	A.D. 141 Feb 2	283°;30	310°;0	26°;30	94

The Apogee of Mercury

We do not know the order in which Ptolemy developed the various planetary theories in the Almagest. However, it is a plausible hypothesis that he set out to construct the theory of Mercury along very much the same lines as that of Venus, beginning by determining the apogee by means of pairs of observations with equal maximum eastern and western elongations. On the other hand he very soon ran into unexpected difficulties which could be resolved only by a model which is at the same time as ingenious – and as complicated – as that of the lunar theory[8]).

In order to determine the apogee of Mercury Ptolemy begins by considering the observations M_9 and M_{10} [IX, 7; Hei 2, 262]. Here the planet has the same maximum eastern and western elongation respectively, viz. 21°;15. Therefore the apsidal line must pass through the point in the middle between the two mean positions $\lambda_{m\odot}(t_9)$ and $\lambda_{m\odot}(t_{10})$, or

$$\frac{\lambda_{m\odot}(t_9) + \lambda_{m\odot}(t_{10})}{2} = \frac{309°;45 + 70°;0}{2} = 9°;52,30$$

In a similar way the pair of observations M_{13} and M_{16}, corresponding to the same maximum elongation of 26°;30, give

$$\frac{\lambda_{m\odot}(t_{13}) + \lambda_{m\odot}(t_{16})}{2} = \frac{70°;30 + 310°;0}{2} = 10°;15$$

The two results are in sufficient agreement to warrant the assumption that the apsidal line of Mercury passes through a point with a longitude of 10°, and therefore also through the opposite point with the longitude 190°.

Now Ptolemy has to decide whether the apogee lies in the direction of $\lambda = 10°$ or $\lambda = 190°$. For this purpose he needs observations in which the mean position of the planet is either one or the other of those two values. No such observations were recorded, so that Ptolemy has to make them himself by means of the *astrolabon*. The observation M_{11} corresponds to a mean position of 189°;15 and M_{12} to 11°;5. These two longitudes are sufficiently near to 190° and 10° respectively to decide the question. Since the maximum elongation at M_{11} is 19°;3 but 23°;15 at M_{12}, it follows that the epicycle looks smaller when the mean position is 190° than when it is 10°. Consequently the apogee is at 190°, so that we have

$$\lambda_a = 190° \tag{10.21}$$

8) This explains why the theory of Mercury has often been construed in a confused and erroneous way by early authors, beginning with Proclus, *Hypotyposis* V, 3 B.

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6) Boelk (p. 18) accused Delambre of dating the observation M_6 to Nabonassar 564, asserting that Herz followed Delambre. It is true that Herz (p. 133) gives this erroneous date, and yet Delambre (II, p. 520) has the correct value Nabonassar 504.

as the first parameter of the model to be found [IX, 8; Hei 2, 270]. This is one of the most faulty of the parameters in the *Almagest*, the error being about 30° (see Gingerich 1971, p. 57).

The Motion of the Apogee

The value $\lambda_a = 190°$ determined above refers to observations made in Ptolemy's own time. In order to decide whether the apogee is fixed or slowly moving relative to the vernal equinox we need some suitable observations of a considerably earlier date. First Ptolemy considers M_2, at which the maximum western elongation is 25°;50 at a mean position of 318°;10 [IX, 7; Hei 2, 264]. There are no records of any observation of an eastern elongation of the same value, but interpolating between M_3 and M_5 Ptolemy is able to deduce [IX, 7; Hei 2, 265] that Mercury has a maximum eastern elongation of 25°;50 in between the two observations at a mean position of 53°;30 (a check on the calculation leads to 53°;19). In the same way as before we conclude that the apsidal line must pass through a point with the longitude

$$\tfrac{1}{2}[318°;10 + 53°;30] = 5°;50$$

which means that the apsidal line has turned eastwards about 10° − 5°;50 = 4°;10 in the interval of approximately 400 years from M_3–M_5 to M_9–M_{10}. This is checked by an independent calculation based upon M_4 and a position found by interpolation between M_6 and M_7. The two results agree very well, and Ptolemy concludes that the apsidal line of Mercury rotates about 4° in 400 years, that is, at the same rate as the fixed stars. Relative to the stars the apsidal line therefore occupies a fixed position.

To-day we know that all apsidal lines are slowly rotating with various angular velocities so that Ptolemy's conclusion is materially false. Nevertheless, it occupies an important place in Ptolemaic astronomy as a whole, although it has a very different status in the various planetary models. As to Mercury Ptolemy must be excused if he regarded the preceding argument as a real proof, based as it is on observations about 400 years apart. In the theory of Venus the corresponding span of time is only a few years and the force of the argument is weaker in proportion. In the superior planets the motion of the apogees has no observational support at all. Here it is only an assumption based on the idea of a certain regularity prevailing in the planetary system at this particular point. The immediate result of this extrapolation is that the simple formula (9.37) is applicable to all the planets.

It is not easy to say whether the varying epistemological status of the motion of the apogees has something to do with the order in which Ptolemy dealt with the various planetary theories, but this may well be the case. For if he had begun with the superior planets he would have had to let the motion of their apogees hover in the air as an arbitrary postulate until, after the Mercury theory had been completed, he could have transformed it into an assumption based upon analogy. This would have been an aesthetic defect of the Great Syntaxis taken as a whole.

The Eccentricity and the Size of the Epicycle

Ptolemy is now able to determine both the radius r of the epicycle, and a constant e which we shall call the eccentricity (but see page 317). The procedure [IX, 8; Hei 2, 269] is exactly as in the theory of Venus. Ptolemy begins by selecting the observations M_{11} and M_{12}, for which we have the mean positions $\lambda_{m\odot}(t_{11}) = 189°;15$ and $\lambda_{m\odot}(_{12}) = 11°;5$. This means that the corresponding positions C_{11} and C_{12} of the epicycle centre are situated very nearly on the apsidal line, with C_{11} near to the apogee. We can now simplify Ptolemy's actual calculations [IX, 9; Hei 2, 274] in the following way.

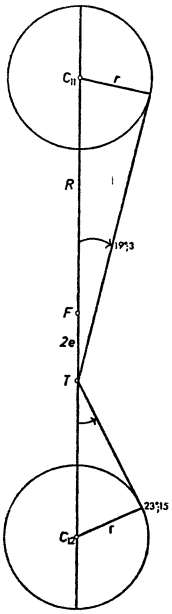

Fig. 10.5

In Figure 10.5 T is the centre of the Earth, and F the middle point between C_{11} and C_{12}. The eccentricity e is defined as half the distance TF. Putting $FC_{11} = FC_{12} = R + e$ we have

$$TC_{11} = R + 3e, \text{ and } TC_{12} = R - e$$

We then have

$$r = (R + 3e) \cdot \sin 19°;3, \text{ and } r = (R - e) \cdot \sin 23°;15$$

with the solutions

$$\frac{e}{R} = \tfrac{1}{2} \cdot \frac{\sin 23°;15 - \sin 19°;3}{\sin 23°;15 + \sin 19°;3} \tag{10.22}$$

and

$$\frac{r}{R} = \frac{4 \sin 19°;3 \cdot \sin 23°;15}{3 \sin 19°;3 + \sin 23°;15} \tag{10.23}$$

Putting $R = 60^p$ we obtain

$$e = 3^p;0 \tag{10.24}$$

and

$$r = 22^p;30 \tag{10.25}$$

The Two Perigees of Mercury

Up to this point the theory of Mercury has been developed along precisely the same lines as that of Venus, but now a very remarkable difference reveals itself with the most curious consequences for the Mercury model. The problem is quite simple. It is to determine the perigee Π of the model.

Now one would expect the perigee of Mercury to be at the opposite point $\lambda_\Pi = 10°$ of the apogee as in all other planets apart from the Moon. But according to Ptolemy this is not so. He calls attention to his own observations M_9 and M_{16} at which the mean positions are 309°;45 and 310° respectively, i.e. very much the same number of degrees. This means that in both cases the epicycle centre has occupied the same position 310° − 190° = 120° East of the apogee. The maximum elongations were at M_9 21°;15 East and at M_{16} 26°;30 West. The sum of these numbers is 47°;45. But M_{12} showed a maximum elongation equal to 23°;15 at $\lambda_{m\odot}(t_{12}) = 11°;5$. Doubling this value we get 46°;30. Therefore Ptolemy concludes that at M_9 and M_{16} the epicycle centre was nearer to the Earth than when it was opposite to the apogee at M_{12}.

This curious lack of symmetry makes Ptolemy examine observations at which the epicycle centre is 120° before the apogee. This is the case for M_{10} and M_{13}, in which the sum of the maximum elongations is 21°;15 + 26°;30 = 47°;45, that is the same as in M_9 and M_{16}. The final conclusion is that Mercury has two perigees, one (Π_1) 120° before ($\lambda_{\Pi 1} = 70°$) and another (Π_2) 120° after ($\lambda_{\Pi 2} = 310°$) the apogee whereas the point opposite to the apogee is no perigee at all [IX, 8; Hei 2, 269]. In Figure 10.6 all the relevant positions of the epicycle are shown in a simplified way.

Such are the data. Unfortunately, there is no doubt that several of the measured values are erroneous. Thus modern computations show that at Ptolemy's time the perigee of Mercury was at $\lambda \approx 50°$ instead of the 10° deduced in the Almagest. Similarly, while the apparent diameter of the epicycle at $\lambda_{m\odot} = 70°$ was, in fact, greater than at $\lambda_{m\odot} = 10°$, there was no corresponding maximum at $\lambda_{m\odot} = 310°$. The reason for these errors may be found partly in the well-known difficulties of observing Mercury at all, partly in the unusually great eccentricity (≈ 0.2) of its Kepler orbit round the Sun. We remember that in the inferior planets it is the motion on the

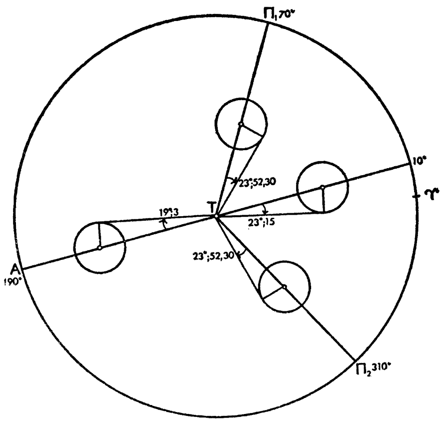

Fig. 10.6

eccentric deferent which reflects the annual motion of the Earth according to the Copernican system, while the Kepler orbit of such a planet corresponds to the epicycle. This means that it is a bad approximation to replace the Kepler ellipse of Mercury by a circular epicycle, because of its great eccentricity. For Venus the similar approximation was much more satisfactory because the eccentricity of the elliptical orbit of Venus is small ($\approx$ 0.0068).

The Mercury Model and its Variables

Ptolemy must have been unaware of these errors, and there is no reason to suppose that he consciously falsified his observational data; otherwise it is a reasonable assumption that he would have falsified them in a way which would have led to a Mercury model of greater simplicity than the complicated structure described in the following. Thus the task set before him was to construct a geometrical model with one apogee and two perigees separated by arcs of 120°. In other words he had to make a model capable of drawing the epicycle centre near to the Earth for mean centrum values of $c_m = 120°$ and 240° instead of 180° as in the other planets.

This was not a completely new difficulty. Already in the lunar theory Ptolemy had encountered a similar problem of drawing the epicycle near to the Earth at points different from the opposite apogee, viz. at the two quadratures. He had solved it by inventing the kinematical device of a movable deferent, the centre D of which rotated upon a small circle concentric with the ecliptic (page 184). The success of his lunar theory may very well have been the reason which caused him to try a similar solution of the Mercury problem. The result was a highly ingenious model, the main features of which are shown in Figure 10.7 [IX, 6; Hei 2, 255].

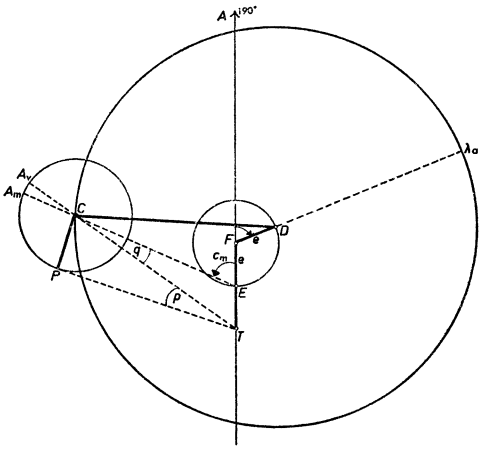

Fig. 10.7

It is seen that the position vector of the planet can be written as a sum

$$\overrightarrow{TP} = \overrightarrow{TF} + \overrightarrow{FD} + \overrightarrow{DC} + \overrightarrow{CP} \tag{10.26}$$

of the following components.

1) $\overrightarrow{TF}$ is a vector with the constant length $2e = 6^p;0$. It defines the apsidal line and points towards the apogee A, forming an angle λ_a with the direction of the vernal equinox. It contains the point of mean motion E (the equant point of Mercury), the exact location of which will be determined in the following section.

A line through E rotating from West to East with constant angular velocity $\omega_\odot$ relative to the apogeum line forms with the latter the

$$\text{angle AEC} = c_m \tag{10.27}$$

called the mean centrum of Mercury. It is a linear function of time which can be expressed here as in the case of the superior planets by (9.15a–b). The mean centrum is connected with the mean longitude $\lambda_m(t)$ by the relation (9.14b), which still holds good.

2) $\overrightarrow{FD}$ is a small vector of the constant length $e = 3^p;0$, rotating from East to West around F with the constant angular velocity $\omega_\odot$ relative to the apsidal line in such a way that the angle AFD is numerically equal to the mean centrum c_m. Thus the deferent centre D will move upon a small circle around F, its position being determined by c_m. This is the *circulus parvus* of the Latin astronomers. Apart from being eccentric it is analogous to the small (concentric) circle carrying the centre of the lunar deferent (see page 184)[9]).

9) According to Dreyer (1906, p. 197 f.) the radius of the small circle is $\frac{1}{21}$ of that of the deferent. This is not in agreement with the Almagest which has TE = EF = 3^p for R = 60^p. Dreyer supports his statement with a reference to the *Planetary Hypotheses*, but here Ptolemy has TE = 3^p and EF = 2;30 (see *Opera Minora*, ed. Heiberg, p. 87). The discrepancy is due to an error in Halma's edition of the *Hypotheses* used by Dreyer.

The vector $\overrightarrow{TD} = \overrightarrow{TF} + \overrightarrow{FD}$ (not shown in the figure) connecting the centre of the Earth with that of the deferent was called the eccentricity vector for both the superior planets and for Venus (page 287 and 303). It is seen that it is not a constant vector in the case of Mercury, where it oscillates about the apsidal line as a mean position with a length varying between e and 3e.

3) $\overrightarrow{DC}$ is the deferent vector which has the constant length $R = 60^p$ and rotates from West to East around the movable centre D. Thus the epicycle centre C moves upon a circle – the eccentric deferent – which has itself a motion in the opposite direction. The exact position of C is determined as the intersection between 1) the moving deferent, and 2) the line through E forming an angle c_m with the apsidal line. As in the theory of Venus the line EC points towards the mean Sun $M_{\odot}{}^*$, like the mean motus line $TM_{\odot}{}^*$ through T and parallel to EC.

4) Finally $\overrightarrow{CP}$ is the epicycle vector which has the constant length $r = 22^p;30$. To describe its motion we must define the mean apogee A_m of the epicycle as the point of the latter furthest away from the centre of mean motion E. Then the

$$\text{angle } A_mCP = a_m \tag{10.28}$$

is called the mean argument. It is a linear function of time, causing the epicycle vector to rotate with the constant angular velocity ω_a relative to A_m.

Besides the mean co-ordinates defined above we shall also need a set of true co-ordinates defined as for Venus or the superior planets. Thus the

$$\text{angle } ATC = c \tag{10.29}$$

is the true centrum. It is connected with the true longitude λ by (9.14a) and measures the angular distance of the epicycle centre C from the apsidal line as seen from the Earth. Introducing the true apogee of the epicycle as the point A_v of the latter furthest away from the Earth T we define the true argument as the

$$\text{angle } A_vCP = a_v \tag{10.30}$$

The mean and true centrum are connected by the relation (9.24), where the equation of centre

$$q(c_m) = \text{angle } ECT \tag{10.31}$$

is the angle under which the line TE is seen from C. Similarly the equation of argument

$$p(a_v, c_m) = \text{angle } CTP \tag{10.32}$$

is the angle under which the epicycle radius is seen from the Earth. From a formal point of view all these definitions are identical with their corresponding numbers in the cases of the superior planets or Venus. This is very convenient since it follows that also the Mercury theory can be summarized by the final formulae (10.12) or (10.13) of the theory of Venus, or their equivalents (9.43) and (9.44) in the theory of the superior planets.

The Centre of Uniform Motion

This model is not completely defined until we have found the exact position of the centre of uniform motion E corresponding to the equant point of the other models. To this purpose Ptolemy considers [IX, 9; Hei 2, 275] the observations M_8 and M_{15} as particularly convenient, since in both of them the mean Sun is very nearly 90° behind the apogee, i.e. the mean centrum is $c_m(t_8) = c_m(t_{15}) \approx 270°$. This gives the

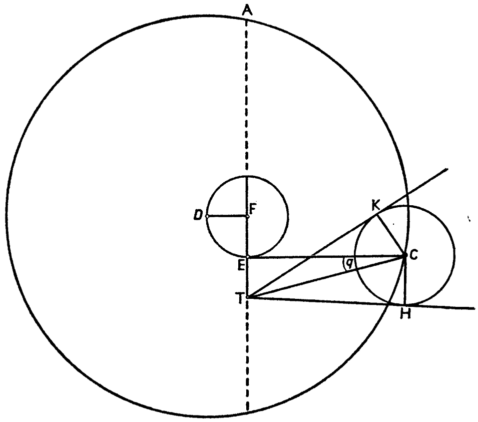

Fig. 10.8

situation shown in Figure 10.8. Here the angle HTK is the sum of the maximum eastern and western elongations, or 26°;15 + 20°;15 = 46°;30. Half this value is the angle CTK = 23°;15. From triangle CTK we then have (by means of $r = 22^p;30$)

$$\rho(c_m) = TC = \frac{22^p;30}{\sin 23°;15}$$

In this model the eccentric anomaly q must be defined as the angle ECT. According to (10.15) it has the value

$$q = \tfrac{1}{2}\{26°;15 - 20°;15\} = 3°;0$$

Since the observations were so selected that triangle CET has a right angle at E, we have

$$TE = TC \sin 3° = \frac{22^p;30}{\sin 23°;15} \cdot \sin 3° = 3^p;0 \tag{10.33}$$

(more exactly 2.983^p). But this means that TE is equal to half the distance TF determined above, i.e. to the eccentricity

$$e = 3^p;0$$

corresponding to R = 60^p. This result is corroborated by another, more approximate method [IX, 9; Hei 2, 278] which we omit here. It means, that E is a point on the small circle around F. Here, as in the Venus theory, there is no question of an iterative method, the value of e being found by direct computation instead of by trial and error.

The Equations of Mercury

We saw above that the formal definitions (10.31) and (10.32) of the equation of centre $q(c_m)$ and the equation of argument $p(a_v, c_m)$ were the same for Mercury as for Venus and the superior planets. But in the Mercury model the distance TE from the Earth to the centre of mean motion was denoted by e, while in the other planets it was 2e. Therefore, we must now investigate whether this influences the actual expres-

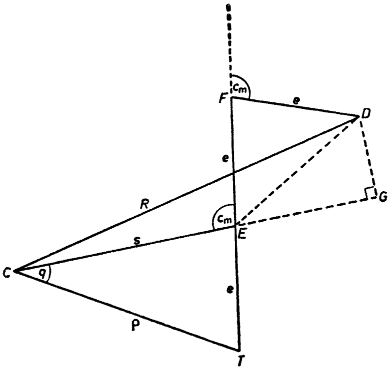

Fig. 10.9

sions of the two equations. In Figure 10.9 the distance EC = $s(c_m)$ of the epicycle centre from E will be a function only of the mean centrum c_m. It appears from triangle DFE that we have

$$\text{angle EDF} = \text{angle DEF} = \frac{c_m}{2}$$

and, accordingly

$$\text{angle DEG} = 180° - \frac{3c_m}{2}$$

where G is the projection of D upon CE.

We have also

$$ED = 2e \cdot \cos \frac{c_m}{2}$$

and consequently

$$GD = ED \sin\left(180^\circ - \frac{3c_m}{2}\right) = 2e \cos\frac{c_m}{2} \cdot \sin\frac{3c_m}{2}$$

$$EG = ED \cos\left(180^\circ - \frac{3c_m}{2}\right) = -2e \cos\frac{c_m}{2} \cdot \cos\frac{3c_m}{2}$$

From triangle CDG follows

$$(s + EG)^2 = R^2 - GD^2$$

or, after reduction

$$s(c_m) = \sqrt{R^2 - \left(2e \cos\frac{c_m}{2} \sin\frac{3c_m}{2}\right)^2} + 2e \cos\frac{c_m}{2} \cos\frac{3c_m}{2}$$

or, finally (cf. Hartner, 1955, p. 109)

$$s(c_m) = \sqrt{R^2 - e^2(\sin c_m + \sin 2c_m)^2} + e(\cos c_m + \cos 2c_m) \qquad (10.34)$$

The expression for $\rho(c_m) = TC$ follows from triangle TEC, viz.

$$\rho(c_m) = \sqrt{s^2 + e^2 + 2\, e\, s \cos c_m} \qquad (10.35)$$

Applying the sine relations to triangle TEC we find for the equation of centre

$$\sin q(c_m) = -\frac{e \sin c_m}{\rho(c_m)} \qquad (10.36)$$

where the sign is so chosen that the usual relation (9.24) is satisfied. It is seen that from a formal point of view (10.36) can be deduced from (9.23) by substituting e for 2e in the latter formula.

It appears from triangle TCP in Figure 10.7 that the expression (9.40) for the equation of argument, viz.

$$\sin p(a_v, c_m) = \frac{r \sin a_v}{\Delta(c_m, a_v)} \qquad (10.37)$$

is still true, the sign being chosen according to the usual convention (page 238). Also the expression (9.39) for the distance $\Delta(c_m, a_v) = TP$ from the Earth to the planet is valid as seen from Figure 10.7. Therefore from a formal point of view the expression of the equation of argument is unchanged. But here as in (10.36) the new procedure for calculating $\rho(c_m)$ by (10.35) and (10.34) will make a difference in the actual computations.

The Behaviour of the Epicycle Centre

As a first test of the model Ptolemy investigates whether it really implies two different perigees Π_1 and Π_2 120° before and after the apogee. This is done by means of Figure

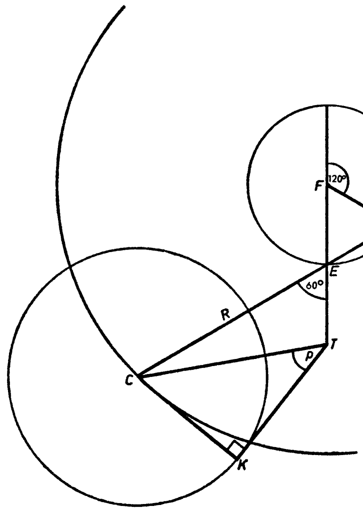

Fig. 10.10

10.10, which illustrates how Ptolemy calculates the apparent radius (or diameter) of the epicycle for a mean centrum value of $c_m = 120°$ [IX, 9; Hei 2, 279].

Because $c_m = 120°$ it follows that triangle FDE is equilateral. Furthermore, since $\sphericalangle$ DFE $= 60°$ and $\sphericalangle$ FEC $= 120°$ we see that DEC is a straight line. This means that we have

$$s(120°) = R - e = 57^p$$

in accordance with (10.34). From (10.35) we can now calculate

$$\rho(120°) = 55^p;34$$

The angular radius of the epicycle is found as the maximum equation of argument corresponding to $c_m = 120°$. The figure shows that we have

$$\sin p_{max}(120°) = \frac{r}{\rho(120°)} = \frac{22^p;30}{55^p;34}$$

from which follows

$$p_{max}(120°) = 23°;53$$

The epicycle is seen under twice this angle, or 47°;46. This is so near to the value 47°;45 deduced from the observations M_{10} and M_{13}, or also M_9 and M_{16}, that Ptolemy is satisfied with the model, as far as the two alleged perigees are concerned.

However, one must admit that here Ptolemy was a little too confident. What his calculation shows is that the model is able to reproduce the apparent size of the epicycle at the critical position $c_m = 120°$. But it did not show that this position really corresponds to a maximum apparent size or a minimum distance of the epicycle centre, which amounts to the same thing. A proof of that presupposes that we investigate the behaviour of the epicycle centre a little more closely.

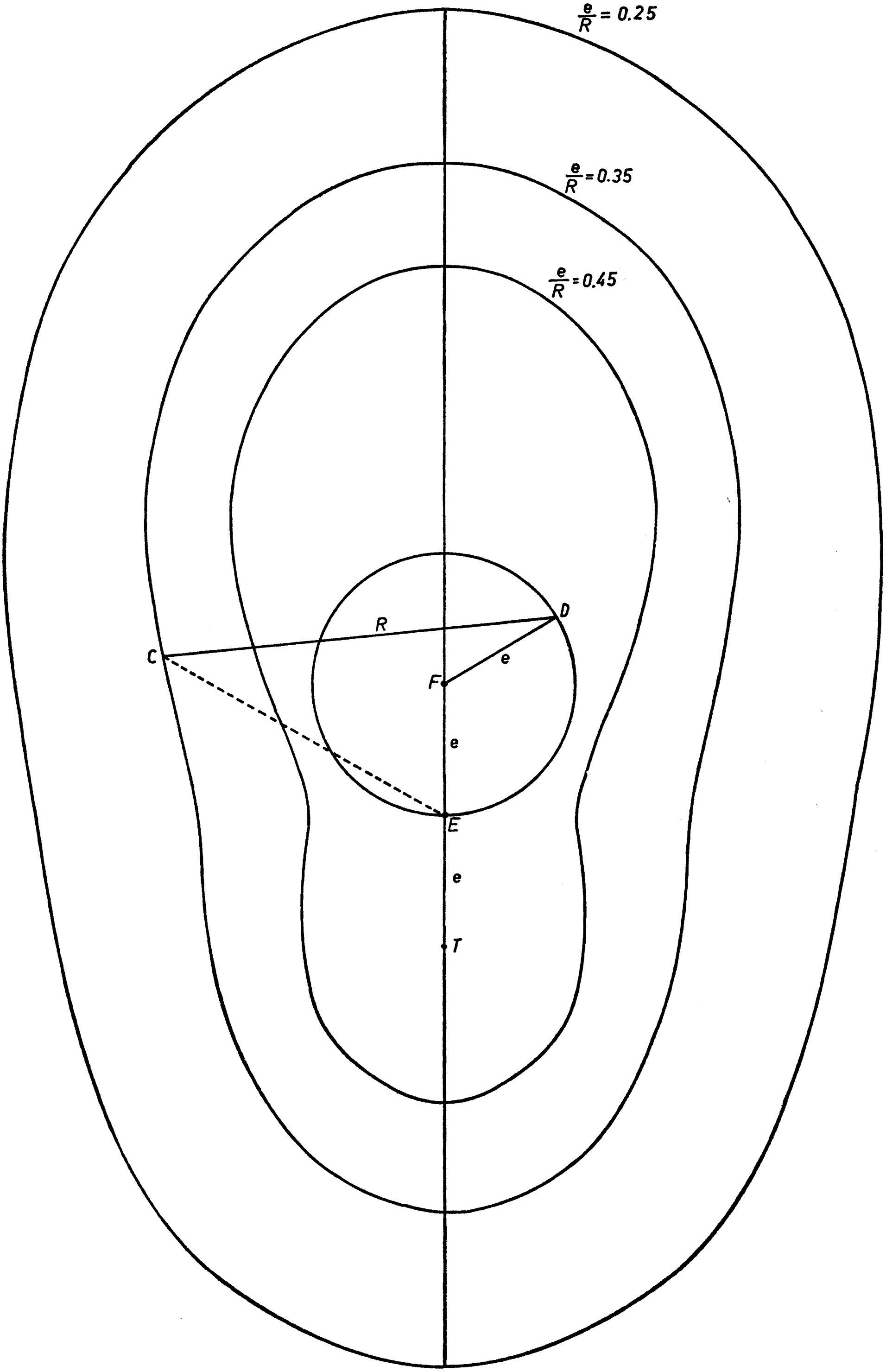

Fig. 10.11

This was first done in a fundamental paper by W. Hartner (1955), who deduced (10.34) and gave a graphic representation of this function in a system of polar co-ordinates with the apsidal line as axis and the point E as origin. For different values of the ratio e/R this resulted in the set of curves shown in Figure 10.11. In general, the curves are pear-shaped, but as e/R diminishes they approach more and more closely to an ellipse. This was proved in detail by Hartner[10]) who also found the minimum of the geocentric distance $\rho(c_m)$ of the epicycle centre. For $e/R = 0.05$ (the Ptolemaic vale) he found (see also Hartner 1964, p. 267, n. 25)

$$\rho_{min} = 55^p;34 \text{ for } c_m \approx 120°;30$$

A new calculation by means of a computer gave the values listed in the following table

c_m	$\frac{\rho(c_m)}{R}$
120°;23	0.9260 08 32 37
120°;24	0.9260 08 26 67
120°;25	0.9260 08 22 07
120°;26	0.9260 08 18 71
120°;27	0.9260 08 16 54
120°;28	0.9260 08 15 57
120°;29	0.9260 08 15 79
120°;30	0.9260 08 17 20
120°;31	0.9260 08 19 81
120°;32	0.9260 08 23 61
120°;33	0.9260 08 28 60
120°;34	0.9260 08 34 78

This shows that as minimum value we have

$$\frac{\rho_{min}}{R} = 0.9260\ 08\ 16$$

which for $R = 60^p$ gives the sexagesimal number

$$\rho_{min} = 55^p;33,38 \qquad (10.38)$$

in agreement with both Hartner's result and the value calculated by Ptolemy. The minimum occurs for a value of the mean centrum inside the interval

$$120°;28 < c_m < 120°;29$$

Performing a numerical differentiation of the function as tabulated above we obtain the value

$$c_m = 120°;28,20 \qquad (10.39)$$

10) This was also indicated by D. J. de Solla Price in *The Equatorie of the Planetis*, Cambridge, 1955, p. 102 which appeared at the same time as Hartner's paper. – Needles to say, Hartner's investigation was concerned with very much more than the derivation of this minimum value.

instead of the 120° uncritically accepted by Ptolemy and usually adopted throughout the Middle Ages.[11])

Corrections and Radices of Mercury

As in the theory of Venus, Ptolemy first makes use of the Mercury model to correct the daily mean motion in anomaly (argument) ω_a. The calculation is based on the observations M_1 and M_{14} and follows exactly the same lines as in the case of the other inferior planet (see page 307). It leads to the value of ω_a already given by (10.4) and used as basis of the tables of mean motion in anomaly [IX, 10; Hei 2, 283].

The observation M_1 is also used to reduce the kinematical parameters to the standard epoch t_0. The result is the following set of radices [IX, 11; Hei 2, 293]

$$\begin{aligned} \lambda_{m\odot}(t_0) &= 330°;45 \\ a_m\ (t_0) &= \ \ 21°;55 \\ \lambda_a\ (t_0) &= 181°;10 \end{aligned} \qquad (10.40a–c)$$

With the determination of these initial values the Mercury model is complete as far as longitudes are concerned. In spite of the intricate kinematics of the model the procedure for calculating the longitude $\lambda(t)$ of Mercury at any given time is the same as for the other planets, and the final longitude formulae (9.43) and (9.44) are still valid. This is a testimony to the fact that although there is no fundamental or intrinsic unity of the planetary theories of the Almagest, Ptolemy nevertheless succeeded in giving to them a formal similarity of great value to the practical calculator.

11) Dr. D. T. Whiteside has kindly provided me with the following analytical treatment of the problem. From (10.35) it follows that the condition $d\rho/dc_m = 0$ for the minimum of $\rho(c_m)$ is equivalent to

$$\frac{ds}{dc_m} = \frac{es \sin c_m}{s + e \cos c_m}$$

The expression (10.34) can be reduced to

$$R^2 = s^2 - 2es(\cos c_m + \cos 2c_m) + 2e^2(1 + \cos c_m)$$

from which follows

$$\frac{ds}{dc_m} = \frac{-es(\sin c_m + 2\sin 2c_m) + e^2 \sin c_m}{s - e(\cos c_m + \cos 2c_m)}$$

Equating the two values of ds/dc_m and dividing by $e \sin c_m \neq 0$ we get

$$2s^2(1 + 2\cos c_m) + 2es\cos^2 c_m = e^2 \cos c_m$$

whence

$$\cos c_m = \frac{-4s^2 + e^2 (\pm) \sqrt{(4s^2 - e^2)^2 - 16es^3}}{4es}$$

Using $e = 0.05 \cdot R$ and the *approximate* value $s \approx s(120°) = 0.9260 \cdot R$ we get

$$c_m = 120°;29,7$$

which is only slightly different from (10.39).

The General Table of Equations

Also in the Mercury theory all lengthy trigonometrical calculations are avoided by means of a general table of equations [XI, 11; Hei 2, 444] enabling us to determine both the equation of the centre and the equation of argument by simple arithmetical operations similar to those described above (page 291). The general arrangement of the 8 columns of this table is exactly as for the other planets. Thus Columns I and II contain general arguments from 6° to 180°, and from 354° backwards to 180° respectively. Column III gives the function $\alpha(c_m)$ and Column IV the function $\delta(c_m)$, the sum of which is the equation of the centre $q(c_m)$ (cf. page 292). The remaining four columns V–VIII are related to the equation of argument $p(a_v, c_m)$, and here we notice some minor differences from what we saw in the cases of the other planets, due to the fact that the model has a moving deferent circle, and that the perigee is supposed to be at $c_m = 120°$. But in his explanation of how this part of the table is constructed [XI, 10; Hei 2, 427] Ptolemy fails to distinguish between Mercury and the other planets. Therefore we must examine the peculiarities of this half of the table exclusively by means of the numerical material.

The Mean Position of the Epicycle Centre

As for the superior planets (page 292) Ptolemy introduces a mean position of the epicycle centre C, defined by the value $c_m°$ of the mean centrum c_m which satisfies the relation

$$\rho(c_m°) = R \tag{10.41}$$

which is formally identical with (9.47). But in the theory of Mercury this relation gives rise to a difficulty. We remember that in planetary models having a fixed deferent the value $c_m°$ could be found by a simple geometrical solution of (10.41) as shown in Figure 9.10, one locus of the mean position of C being the fixed deferent and another a circle of radius R drawn around T as centre.

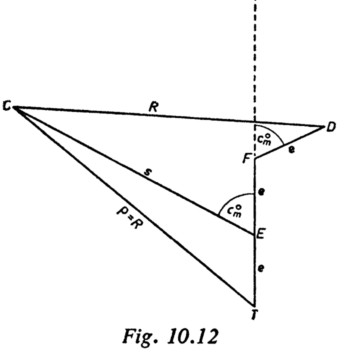

Fig. 10.12

In the case of Mercury no such construction is possible. Also here one locus of C is a circle around T with the radius R (see Figure 10.12). But as a second locus of C we cannot use the deferent since its (movable) position depends on the very parameter $c_m°$ we are trying to determine. Thus a geometrical solution of (10.41) is impossible. An analytical treatment of the problem implies that we put $\rho = R$ in (10.35) and solve this equation and (10.34) with respect to $c_m°$ and s. This is impossible from a practical point of view, considering the general level of algebra and trigonometry at Ptolemy's disposal. Thus the most probable conclusion is that the value of $c_m°$ implied in the Almagest – but nowhere explicitly quoted – was found by trial and error.

We have solved the problem by means of a computer, determining the values of the function

$$\frac{\rho(c_m)}{R}$$

which for $c_m = c_m°$ must be equal to unity. The results appear from the following short table

c_m	$\frac{\rho(c_m)}{R}$
66°	1.00482
67°	1.00203
68°	0.99928
69°	0.99656

By linear interpolation this leads to the following approximate value

$$c_m° = 67°;45 \tag{10.42}$$

It will be shown below (page 328) that this agrees fairly well with the value implied in the Almagest.

The fact that the perigee of the Mercury model is at $c_m \approx 120°$ and not opposite to the apogee implies that we must here define 4 standard positions of the epicycle centre instead of 3 as with the superior planets (page 292 f.), viz.

1) $c_m = 0°$	maximum distance	$TC = R + 3e$ (apogee)
2) $c_m = c_m°$	mean position	$TC = R$
3) $c_m = 120°$	minimum distance (perigee)	
4) $c_m = 180°$	intermediate distance	$TC = R - e$

The importance of these points will appear in the following section.

The Equation of Argument

In the equation of argument $p(a_v, c_m)$ the independent variable a_v is strong while c_m is weak. It is clear that for any constant value of a_v the equation must (considered as a

function of c_m) have a minimum at the apogee, a greater value at the mean position, a maximum at the perigee, and a somewhat smaller value at the position $c_m = 180°$ opposite to the apogee. To account for this variation Ptolemy applies an adaptation of his standard interpolation method (page 86). First we define three functions of the strong variable a_v

$$p_1(a_v) = p(a_v, 0°) \tag{10.43}$$
$$p_0(a_v) = p(a_v, c_m°) \tag{10.44}$$
$$p_2(a_v) = p(a_v, 120°) \tag{10.45}$$

representing the equation of argument as a function of a_v at each of the first three standard positions respectively and analogous to the functions (9.51), (9.46), and (9.49) in the theory of the superior planets; the only difference is that (10.45) is taken at the point $c_m = 120°$ while (9.49) refers to $c_m = 180°$. As before, Ptolemy considers $p_0(a_v)$ as the principal function and tabulates it in Column VI of the table of equations. Likewise the difference

$$p_0(a_v) - p_1(a_v) \tag{10.46}$$

is tabulated in Column V, and the difference

$$p_2(a_v) - p_0(a_v) \tag{10.47}$$

is tabulated in Column VII. Then he states (without proof) that the equation of argument $p(a_v, c_m)$ for any value of a_v and c_m can be found by a procedure corresponding to one of the following formulae (cf. 3.57):

a) $0° \leqslant c_m < c_m°$

In this first interval we have

$$p(a_v, c_m) = p_0(a_v) + [p_0 - p_1] \cdot \frac{f'(c_m)}{60} \tag{10.48}$$

b) $c_m° \leqslant c_m < 120°$

In the second interval the formula is

$$p(a_v, c_m) = p_0(a_v) + [p_2 - p_0] \cdot \frac{f''(c_m)}{60} \tag{10.49}$$

c) $120° \leqslant c_m \leqslant 180°$

This third interval between the third and the fourth standard positions is peculiar to the theory of Mercury. Here a direct calculation of the distance $\rho(c_m)$ of the epicycle centre at these two positions shows that

$$\rho(120°) = 0.9260 \cdot R$$
$$\rho(180°) = 0.9500 \cdot R$$

(cf. page 323)

This means that throughout this interval the equation of argument must (for the same value of a_v) be greater than at the mean position where $\rho(c_m°) = R$. It must then be found by adding something to $p_3(a_v)$, and Ptolemy assumes that it can be expressed

simply by (10.49). This means that here the interpolation formulae (3.57) and (3.58) are applied to an interval in which the function is not monotone.

To investigate the construction of the interpolation functions in the formulae above, we calculate the maximum values max $p(c_m)$ of the equation of argument at the four standard positions, considering it as a function of a_v only. The result

$$\sin \max p(c_m) = \frac{r}{\rho(c_m)} \tag{10.50}$$

follows from a simple geometrical consideration, max $p(c_m)$ being the angle between TC and the tangent from T to the epicycle. The resulting values are

$$\begin{aligned} \max p(0^\circ) &= 19^\circ;\ 2 \\ \max p(c_m{}^\circ) &= 22^\circ;\ 2 \\ \max p(120^\circ) &= 23^\circ;53 \\ \max p(180^\circ) &= 23^\circ;15 \end{aligned}$$

in agreement with the Almagest [XI, 10; Hei 2, 443]. Calculating also max $p(c_m)$ for other values of c_m listed in Columns I and II it appears that the two interpolation functions are defined by (cf. 3.58)

$$1)\ f'(c_m) = 60\,\frac{\max p(c_m) - \max p(c_m{}^\circ)}{\max p(c_m{}^\circ) - \max p(0^\circ)} \tag{10.51}$$

for $0^\circ \leqslant c_m \leqslant c_m{}^\circ$

$$2)\ f''(c_m) = 60\,\frac{\max p(c_m) - \max p(c_m{}^\circ)}{\max p(120^\circ) - \max p(c_m{}^\circ)} \tag{10.52}$$

for $c_m{}^\circ \leqslant c_m \leqslant 180^\circ$

These functions are tabulated consecutively in Column VIII of the table of equations (cf. page 294). They may be considered as separate sections of a single continuous interpolation function $f(c_m)$, which increases from a minimal value $f(0^\circ) = -60$ to $f(c_m{}^\circ) = 0$ at the mean distance. Then it increases to $f(120^\circ) = +60$ at the perigee, only to decrease again to $f(180^\circ) = 39;28$ at the point opposite to the apogee. This enables us to check the value (10.42) of $c_m{}^\circ$. We find in the table

$$\begin{aligned} f(66^\circ) &= (-)2;30 \\ f(72^\circ) &= (+)9;14 \end{aligned}$$

A linear interpolation shows that $f = 0$ for

$$c_m{}^\circ = 67^\circ;13$$

in tolerable agreement with (10.42).

The Mercury theory was certainly much less accurate than the theories of the other planets, but it is difficult to say how good or bad it was. Ephemerides calculated by O. Gingerich (1971) for two periods 1300–1301 and 1555–1556 and based on the Alphonsine tables (which have Ptolemaic parameters) indicate errors in longitude amounting to about 12° at the (invisible) inferior conjunctions.

CHAPTER 11

Retrograde Motions
and Maximum Elongations

Introduction

In the previous Chapters 5, 6, 9, and 10 we have seen how Ptolemy solved the problem of determining the ecliptic longitude $\lambda(t)$ of the Sun, Moon, and planets at any given instant of time t. The solution was given in the form of a number of fairly similar standard procedures which we have summarized in the final formulae (5.32–33) of the solar theory, (6.8) of the lunar theory, and (9.43–44) together with (10.12) in the case of the five planets. This was an intellectual feat of the highest importance for the development of astronomy. For not only did Ptolemy complete what his predecessors had begun, but his achievement meant that the purpose of planetary theory became definitively fixed in a new direction.

As mentioned above (page 267), pre-Greek astronomy was concerned mainly with procedures for determining synodic phenomena, such as oppositions, first and last stationary points, first and last visibility of the planets, and eclipses of the Sun and the Moon. What happened in between these events was considered less important, so that the first theoretical astronomers satisfied themselves with a rather limited programme.

Now a new situation had arisen. Ptolemy's planetary theories had crowned the work begun by Hipparchus, and theoretical astronomy had become able to deal with planetary motion in a very general way, and to predict longitudes for any given time, whether or not the time in question marked a synodic event. Therefore, the synodic phenomena lost much of their theoretical importance. Nevertheless, they still had to be dealt with for a number of reasons. Thus eclipses were important both for astrological reasons and as possible means of finding geographical longitudes, wherefore Ptolemy very carefully developed a general theory of eclipses in Book VI of the Almagest (see Chapter 7).

On the other hand both retrograde motions and visibility periods lost their character of being fundamental astronomical phenomena, and Ptolemy had to deal with them only for reasons of a secondary nature. Perhaps the ancient point of view still prevailed to such an extent that it was impossible not to answer precisely those questions which weighed most heavily on the minds of traditional astronomers.

In the preceding chapters we have seen a number of examples of how the new procedures were applied to practical calculations. Many more examples are found in the Almagest, and an even larger number of calculations of longitudes must have been performed by Ptolemy during the ten or fifteen years in which the Almagest was

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_11,

under way. Although the work contains no general discussion of the validity or accuracy of the new methods, there is no doubt that Ptolemy had become convinced of their satisfactory character, realizing that all phenomena connected with the motion in longitude could in principle be described by the formulae or procedures quoted above – provided only that suitable mathematical tools for handling them were at hand.

Now, considering the success of the theory of longitude in general, it is an ironical fact that the latter requirement (of proper mathematical tools) was missing when the theory was to be applied to some of the most conspicuous among the phenomena it was designed to explain.

We know already (page 264) that all the planets apart from the Sun and the Moon sometimes abandon their direct (eastward) motion in order to become retrograde, i.e. to move from East to West contrary to their usual course relative to the fixed stars. In between the direct and the retrograde phases there will be two short intervals of time at which the planet has no apparent movement at all. We define the *first stationary point* S_1 (or the *first station*) as the point on the ecliptic where the direct motion stops, and the *second station* S_2 as the point where it is resumed again. The intermediate arc S_1S_2 is called the *retrograde arc*. We know that somewhere about the middle of S_1S_2 a superior planet will be in opposition (page 263), while an inferior planet is in (lower) conjunction with the Sun.

Considering the longitude of a planet as a function $\lambda(t)$ of time we can characterize

direct motion	by $\dot{\lambda}(t) > 0$
retrograde »	by $\dot{\lambda}(t) < 0$

and the

$$\text{stationary points} \quad \text{by } \dot{\lambda}(t) = 0 \tag{11.1}$$

with the further conditions $\ddot{\lambda}(t) < 0$ for S_1 and $\ddot{\lambda}(t) > 0$ for S_2. In principle the times of the first and second station can be found by solving (11.1) with respect to t. But this formulation of the problem is, of course, impossible in Ptolemaic astronomy. First, the Almagest has no explicit expression for the function $\lambda(t)$, but only a procedure for calculating its values. Second, the derivative $\lambda(t)$ is a modern mathematical concept foreign to ancient mathematics notwithstanding the fact that Ptolemy was able to determine numerical values of instantaneous angular velocities, as we have seen in Chapter 3. Thus a general determination of the stationary points is not feasible. Since the calculation of synodic phenomena in general was one of the principal aims of traditional astronomy, this was a very serious defect of the theory.

It would have been possible to cope with the problem by computing an ephemeris for each planet. In fact, a table of actual planetary positions covering a sufficiently large span of time would immediately reveal the times and positions of S_1 and S_2 but at the cost of a tremendous amount of numerical work, from which Ptolemy thought it best to refrain. Instead he fell back on an earlier method which – although

it has an approximative character – enabled him to describe these phenomena. This is the subject matter of Book XII of the Almagest[1]).

The Theorem of Apollonius

Already at the beginning Ptolemy admits that he was not the first to deal with this problem, but that it had been investigated by other mathematicians including Apollonius of Perga [XII, 1; Hei 2, 450]. In fact, the fundamental theorem upon which the whole theory is founded is explicitly ascribed to Apollonius [ibid.; Hei 2, 456], of whom we already know that he examined the properties of a model with an epicycle moving upon a concentric deferent (page 137). In terms of such a model the theorem can be stated as follows (see Figure 11.1): Let TSV be a secant from the centre of the Earth to the epicycle, intersecting the latter at the points S and V so that S is nearest to T, and let U be the middle point of SV. Then S will be a stationary point if the ratio SU:TS is equal to the ratio between the velocity of the epicycle and the velocity of the planet [XII, 1; Hei 2, 450–51].

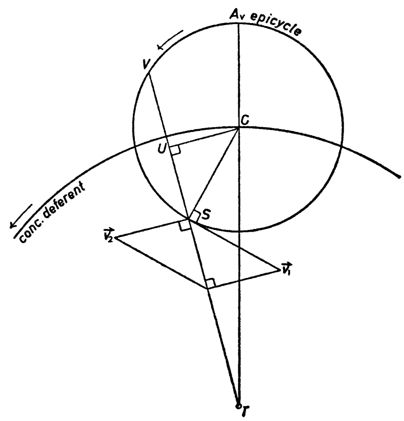

Fig. 11.1

What this means is not quite clear since Ptolemy does not define the two velocities. How the theorem is to be understood appears from a modern proof[2]), based on the parallelogram of velocities. The figure shows the epicycle with the planet at S. If the epicycle were fixed upon the deferent, then S would have a velocity vector $\vec{v_1}$ perpendicular to SC with the length

$$|\vec{v_1}| = r \cdot \dot{a}_v$$

1) There is no great amount of secondary literature to this chapter. The reader is referred to Delambre (1817, II, 381–392), Herz (1887, 47–50 and 159–161), and O. Neugebauer (1959 a).
2) The following proof is taken from B. L. van der Waerden, *Science Awakening*, p. 238.

where r is the radius of the epicycle, and $\dot{a}_V$ is the angular velocity of CS relative to the true apogee A_V (i.e. the time derivative of the true anomaly a_V).

On the other hand, if S were fixed on the epicycle relative to A_V, but the whole epicycle moved around the deferent, then S would have a velocity $\vec{v_2}$ perpendicular to TS and with the magnitude

$$|\vec{v_2}| = TS \cdot \dot{\lambda}_c$$

where $\dot{\lambda}$ is the angular velocity of the epicycle centre with respect to the Earth, λ_c being the ecliptic longitude of C which – in the theory of Apollonius – is equal to the mean longitude of the planet.

The condition that S is a stationary point is that the vector $\vec{v_1} + \vec{v_2}$ points towards T. From the similar triangles in the figure we have

$$\frac{|\vec{v_1}|}{r} = \frac{|\vec{v_2}|}{\frac{1}{2}SV}$$

or

$$\frac{r \cdot \dot{a}_V}{r} = \frac{TS \cdot \dot{\lambda}_c}{\frac{1}{2}SV}$$

or again

$$\frac{\frac{1}{2}SV}{TS} = \frac{\dot{\lambda}_c}{\dot{a}_V} \tag{11.2}$$

in agreement with Ptolemy's statement, if the 'velocity of the epicycle' is interpreted as $\dot{\lambda}_c$, and the 'velocity of the planet' as $\dot{a}_V$. With the usual notation $\omega_t = \dot{\lambda}_c$ and $\omega_a = \dot{a}_V$ we can write (11.2) as

$$\frac{\frac{1}{2}SV}{TS} = \frac{\omega_t}{\omega_a} \tag{11.2 a}$$

In the following we shall see how Apollonius' theorem is proved in the Almagest, and how it is applied to numerical calculations for each individual planet.

Three Auxiliary Theorems

The general proof is preceded by three lemmas of a purely geometrical nature which we shall state and prove exactly as in the Almagest.

Lemma I [XII, 1; Hei 2, 452]. – On the epicycle (see Figure 11.2) we mark the equal arcs $\Pi_V S_1$ and $\Pi_V S_2$ to each side of the true perigee Π_V and draw the secants TS_1V_1 and TS_2V_2. The lines S_1V_2 and S_2V_1 will intersect at the point K on the diameter $\Pi_V A_V$. The angle $A_V V_2 \Pi_V$ will be 90°. A line through Π_V perpendicular to $V_2\Pi_V$ will

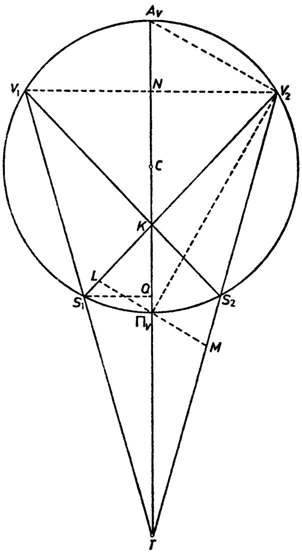

Fig. 11.2

be the base of the isosceles triangle LMV_2 in which $L\Pi_v = \Pi_vM$. We then have the identity

$$\frac{A_vV_2}{L\Pi_v} = \frac{A_vV_2}{\Pi_vM}$$

The similar triangles $A_vV_2T \sim \Pi_vMT$ give

$$\frac{A_vV_2}{\Pi_vM} = \frac{TA_v}{T\Pi_v}$$

In the same way we obtain from $A_vV_2K \sim L\Pi_vK$

$$\frac{A_vV_2}{\Pi_vL} = \frac{KA_v}{K\Pi_v}$$

Combining the three proportions we find

$$\frac{TA_v}{T\Pi_v} = \frac{KA_v}{K\Pi_v} \tag{11.3}$$

as the expression of Lemma I. This consequently states that the diameter $A_v\Pi_v$ is divided internally and externally in the same ratio, which is easily seen to be equal to

$$\frac{TA_v}{T\Pi_v} = \frac{KA_v}{K\Pi_v} = \frac{R+r}{R-r} = \frac{\Delta_{max}}{\Delta_{min}} \tag{11.4}$$

where $\Delta_{max} = R + r$ and $\Delta_{min} = R - r$ are the maximum and minimum distances of the planet from the observer. This relation has two important corollaries.

First, it follows immediately from (11.4) that the position of K is completely determined by the position of T and thus it is independent of the size of the arcs $\Pi_v S_1 = \Pi_v S_2$; this is implied in what Ptolemy says, but not stated in so many words.

Second, denoting R by CT we have

$$TA_v = CT + r$$
$$T\Pi_v = CT - r$$

and also

$$KA_v = r + CK$$
$$K\Pi_v = r - CK$$

We can then write (11.3) in the form

$$\frac{CT + r}{CT - r} = \frac{r + CK}{r - CK} \tag{11.5}$$

which is reduced to

$$CK \cdot CT = r^2 \tag{11.6}$$

This relation shows that T and K are corresponding points of the inversion defined by the circle with centre C and radius r, i.e. the epicycle. This property will be used later (page 340).

Lemma II [XII, 2; Hei 2, 454]. – The second lemma states that

$$\frac{TV_1}{TS_1} = \frac{KV_1}{KS_1} \left[= \frac{KV_1}{KS_2} \right] \tag{11.7}$$

The proof follows from Figure 11.2 by the similar triangles $V_1TN \sim S_1QT$, where N and Q are the points of intersection between the diameter $A_v\Pi_v$ and the lines perpendicular to it through V_1 and S_1 respectively. That the third ratio in (11.7) is equal to the second follows from $KS_1 = KS_2$.

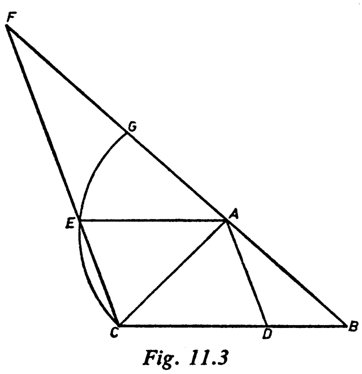

Fig. 11.3

Lemma III [XII, 1; Hei 2, 456]. – This lemma is concerned with a triangle ABC in which BC > AC (see Figure 11.3). On BC lies D so that CD $\geqslant$ AC. We can then prove that

$$\frac{CD}{DB} > \frac{\text{angle B}}{\text{angle C}} \tag{11.8}$$

Ptolemy ascribes the following proof to Apollonius himself: We determine the point E so that ADCE is a parallelogram. CE and BA are supposed to intersect in F. Then a circle is drawn around A as centre with radius AE = CD $\geqslant$ AC. Let us assume at first that CD = AC, in which case the circle will pass through C, intersecting BF in G.

Since

$$\text{triangle AEF} > \text{sector AEG}$$

and

$$\text{triangle AEC} < \text{sector AEC}$$

it follows that

$$\frac{\text{triangle AEF}}{\text{triangle AEC}} > \frac{\text{sector AEG}}{\text{sector AEC}}$$

which is seen to be equivalent to

$$\frac{EF}{EC} > \frac{\text{angle B}}{\text{angle C}}$$

Now AE is parallel to BC, which implies that

$$\frac{EF}{EC} = \frac{AF}{AB}$$

Furthermore AD is parallel to CF so that

$$\frac{AF}{AB} = \frac{CD}{DB}$$

Combining the three latter relations we get (11.8).

Next we ought to prove our lemma in the case of CD > AC, but here Ptolemy satisfies himself with the brief remark that the left hand side of the inequality (11.8) will be even greater than before.

Ptolemy's Proof of Apollonius' Theorem

We are now prepared to tackle the general proof of the fundamental theorem (11.1) and shall here follow the same course as Ptolemy in the Almagest [XII, 1; Hei 2, 458].

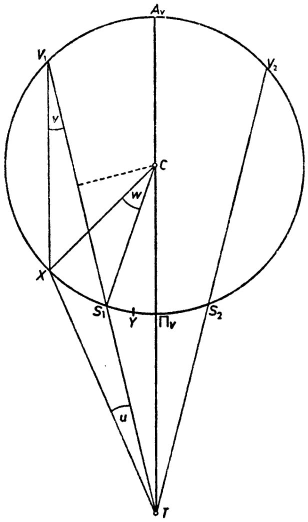

Fig. 11.4

In Figure 11.4 the circle represents an epicycle, and the outside point T the centre of the Earth, or, very nearly, the position of the observer. We must now distinguish between two different cases.

A. First, let us assume that

$$\frac{C\Pi_V}{\Pi_V T} = \frac{r}{R - r} > \frac{\dot{\lambda}_c}{\dot{a}_V} \tag{11.9}$$

where $\dot{a}_V$ and $\dot{\lambda}_c$ as before are the angular velocities of the planet on the epicycle, and of the epicycle centre on the concentric deferent, respectively. Writing this assumption in the form

$$r \cdot \dot{a}_V > (R - r) \cdot \dot{\lambda}_c \tag{11.10}$$

it is seen to imply that the linear velocity with which the planet is carried along the epicycle is greater than the velocity with which the perigee of the epicycle is carried along the deferent. Therefore the planet will be retrograde around the perigee of the epicycle.

B. Second, under the assumption that

$$\frac{C\Pi_V}{\Pi_V T} = \frac{r}{R - r} < \frac{\dot{\lambda}_c}{\dot{a}_V} \tag{11.11}$$

or

$$r \cdot \dot{a}_V < (R - r) \cdot \dot{\lambda}_c \tag{11.12}$$

it follows that the planet is not retrograde at the perigee. Since (11.12) implies

$$r \cdot \dot{a}_V < (R + r) \cdot \dot{\lambda}_C \tag{11.13}$$

it also follows that it is not retrograde at the apogee either. In this case there will accordingly be no retrograde motion at all [XII, 2; Hei 2, 463].

Consequently we must return to case A. We draw a secant TSV to the epicycle in such a way that TS < TV. We know from Euclid III, 8 that when this line is turned from the tangential position towards the centre C the distance TS will decrease but TV will increase. More precisely we have

$$1 < \frac{TV}{TS} < \frac{TA_V}{T\Pi_V} = \frac{R + r}{R - r}$$

from which follows

$$0 < \frac{TV - TS}{TS} < \frac{2r}{R - r}$$

or

$$0 < \frac{\frac{1}{2} SV}{TS} < \frac{r}{R - r} \tag{11.14}$$

Now (11.9) can be written in the form

$$0 < \frac{\dot{\lambda}_C}{\dot{a}_V} < \frac{r}{R - r} \tag{11.15}$$

Therefore, inside this interval there must be a position TS_1V_1 of the secant where we have

$$\frac{\frac{1}{2}S_1V_1}{TS_1} = \frac{\dot{\lambda}_C}{\dot{a}_V} \tag{11.16}$$

Let this be the position actually shown in Figure 11.4, and let TS_2V_2 be another secant symmetrical with TS_1V_1 with respect to TC.

According to (11.2) we shall now prove that S_1 and S_2 are stationary points. In the case of S_1 this is done by proving that at an arbitrary point X before S_1 the planet is direct, but indirect at an arbitrary point Y after S_1, provided that X and Y are not too distant from S_1. Let us consider first the point X from which the planet moves to S_1 during a certain interval of time Δt. To prove that this motion is direct as seen from T we must find out how the line of sight from T to the planet behaves during the time Δt. It is seen from the figure that because of the motion of the planet on the epicycle the line of sight will turn a certain angle $XTS_1 = u$ in the retrograde direction (westwards); but this will be more or less compensated by the motion of the epicycle itself on the deferent, as a result of which the line of sight will move a certain angle in the opposite direction (eastwards). These two components must be examined separately.

In the triangle V_1XT we have

$$V_1S_1 > V_1X$$

according to Euclid III,15. This means that we can apply Lemma III above the result that

$$\frac{S_1V_1}{TS_1} > \frac{u}{v}$$

where $v =$ angle XV_1S_1. Consequently we have

$$\frac{\frac{1}{2}S_1V_1}{TS_1} > \frac{u}{2v} \tag{11.17}$$

But 2v is equal to the angle $XCS_1 = w$ through which the planet has turned about C during the time Δt, given by

$$w = \dot{a}_V \cdot \Delta t \tag{11.18}$$

Combining (11.2) and (11.17) we have

$$\frac{\dot{\lambda}_c}{\dot{a}_V} > \frac{u}{w} \tag{11.19}$$

Next let us define an angle w′ by the relation

$$\frac{w'}{w} = \frac{\dot{\lambda}_c}{\dot{a}_V} \tag{11.20}$$

It follows from (11.19) and (11.20) that

$$w' > u \tag{11.21}$$

The direct motion of the line of sight due to the eastward movement of the epicycle on the deferent around T is determined by $\dot{\lambda}_c \cdot \Delta t$. From (11.18) and (11.20) it is found to be

$$\dot{\lambda}_c \cdot \Delta t = \dot{a}_V \cdot \Delta t \cdot \frac{w'}{w} = w' \tag{11.22}$$

Since $w' > u$ according to (11.21), it follows that the combined effect will be an eastward displacement of the line of sight. In other words, the planet performs a direct motion from X to S_1.

In very much the same way it can be proved that if Y is a point between S_1 and the perigee Π_V then the planet will be retrograde upon the arc S_1Y. It follows that S_1 separates the direct from the retrograde motion, and that it is the first station of the planet. It is easily seen that the second station S_2, where the motion changes from being retrograde to being direct again, must be a point symmetrical with S_1 with respect to the line TC.

The Equivalent Eccentric Model

The proof given above was formulated in terms of an epicycle model with a concentric deferent, similar to the First Lunar Model (see page 167) in all respects apart from the fact that, unlike the Moon, the planet has a direct motion on the epicycle. Now it will be remembered that the first Lunar Model was equivalent to a model with a moving eccentric deferent, but no epicycle (see page 166, cf. Figure 6.1). A similar equivalence is established in the theory of retrograde motions, where Ptolemy introduces a model with no epicycle, but an eccentric circle, the centre of which *moves about the centre of the ecliptic in the direction of the Signs, and with the velocity of the Sun, while the planet moves upon the eccentric about the centre of the latter, contrary to the direction of the Signs, and with its anomalistic velocity* [XII, 1; Hei 2, 451].

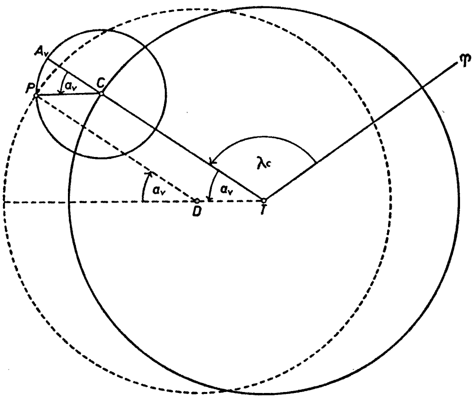

Fig. 11.5

That the two models are equivalent appears from Figure 11.5. Here the epicycle model (which we shall refer to as Model I) is characterized by the position vector

$$\overrightarrow{TP} = \overrightarrow{TC} + \overrightarrow{CP}$$

while the eccentric model (Model II) is characterized by

$$\overrightarrow{TP} = \overrightarrow{TD} + \overrightarrow{DP}$$

It is seen that the two models are equivalent and lead to the same positions of the planet if the quadrilateral TCPD is a parallelogram. This implies

1) that $\overrightarrow{TD}$ has the length r and rotates about T with the angular velocity $\omega_a = \dot{a}_v$ towards the East relative to $\overrightarrow{TC}$. Since the latter vector has the angular velocity $\omega_t = \dot{\lambda}_c$ relative to the vernal equinox, it follows that the resultant angular velocity of TD relative to the latter point is $(\omega_t + \omega_a)$, which is equal to the mean motion $\omega_\odot$ of the Sun according to the relation (9.10) valid for the superior planets. In fact, Ptolemy says

that the eccentric model is applicable only to those planets which can be in opposition to the Sun;

2) that $\overrightarrow{DP}$ has the length R and rotates about D towards the West with the angular velocity ω_a.

Thus Ptolemy is right in asserting the equivalence of the two models. There remains the question of why he bothers to introduce an eccentric model of this kind. It is very dissimilar to the model he actually chose for describing the motion of the superior planet, with a fixed eccentric deferent to account for the first anomaly, and an epicycle to account for the second. Perhaps the most plausible reason is that the following proof of Apollonius' theorem in terms of the eccentric model seems to be a piece of original work, which Ptolemy found it natural to present inside his general exposition of planetary theory although it is without any immediate use[3]).

Apollonius' Theorem in the Eccentric Model

We shall now see how Ptolemy is able to state the theorem of Apollonius in the eccentric Model II described in the previous section. At first sight his treatment appears to be a little bewildering since he makes use of the same figure as in the epicyclic Model I (cf. Figures 11.2 and 11.4). The reason why this is possible is that the epicycle of Model I has one important property in common with the eccentric of Model II, viz. that of producing the same value of the ratio $\Delta_{max}/\Delta_{min}$ between the maximum and minimum distances of the planet from the observer (or rather, the centre of the Earth). Now in Model I the circle in the figure represents an epicycle while the exterior point T represents the position of the observer. If the same circle is going to represent the eccentric of Model II, it follows that now the position of the observer must be represented by an interior point so situated that the ratio $\Delta_{max}/\Delta_{min}$ is the same as before. We know from (11.4) that this is the case for the point K, and for that point only. In other words, the transition from Model I to Model II can be made without changing the diagram if the centre of the Earth (that is of the ecliptic) is displaced from T to K[4]).

3) G. J. Toomer says (1970, p. 190) that *The procedure is worthy of Apollonius, and is indeed a particular case of the pole-polar relationship treated in Conics III, 37. But Ptolemy (who of all ancient authors is most inclined to give credit where it is due) seems to introduce this device as his own, and return to Apollonius only later.*

4) Delambre (1817, II, 382 f.) noticed that the same figure may be applied to both the eccentric and the epicyclic model, but nobody seems to have realized that the transformation involved is a geometrical inversion until O. Neugebauer published his paper on Apollonius (1959 a) from which we quote: *The most conspicuous feature of this discussion is the method of transforming eccentric and epicyclic models into each other by means of an inversion on a fixed circle which serves as eccenter or as epicycle. It seems to have escaped attention, however, that Ptolemy refers to exactly the same method also in Book IV, Chapter 6, on the occasion of his discussion of the determination of the epicycle radius (or of the eccentricity) of the simple lunar theory. I have no doubt that also this section belongs to Apollonius.*

Now Ptolemy states the theorem as follows (cf. Figure 11.2): Inside the eccentric circle a chord S_2KV_1 is drawn through the centre K of the ecliptic in such a way that

$$\frac{\frac{1}{2}S_2V_1}{KS_2} = \frac{\omega_\odot}{\omega_a} \tag{11.23}$$

Then S_2 will be one of the stationary points [XII, 1; Hei 2, 451].

The proof is geometrical and follows the same general lines as in Model I. We shall not repeat it here, but instead prove the equivalence of (11.23) and (11.2a) in a more formal way. By (9.10) we can write (11.23) in the form

$$\frac{\frac{1}{2}(KS_2 + KV_1)}{KS_2} = \frac{\omega_t + \omega_a}{\omega_a} \tag{11.24}$$

or

$$1 + \frac{KV_1}{KS_2} = 2 \cdot \frac{\omega_t}{\omega_a} + 2$$

Using (11.2a) and $KS_2 = KS_1$ we obtain

$$\frac{KV_1}{KS_1} = \frac{S_1V_1}{TS_1} + 1$$

or

$$\frac{KV_1}{KS_1} = \frac{TV_1}{TS_1}$$

which is true according to (11.7). This identity completes the proof.

The Generalization of Apollonius' Theorem

The theorem of Apollonius is stated in terms of the Apollonian model in which the epicycle moves upon a concentric deferent. Consequently it cannot be applied without further ado to the Ptolemaic model in which the epicycle rides on an eccentric deferent. It is also easy to see that the simple Apollonian model is unable to account for the observed phenomena. For in this model both $\dot{\lambda}_c$ and $\dot{a}_v$ are constants, for which reason (11.2) will lead to the conclusion that all retrograde arcs of a given planet are of equal length; but this is contrary to experience. Thus the theorem has to be generalized in such a way that it can be adapted to the Ptolemaic models. This aim was achieved by Ptolemy in a way which must be deemed very ingenious in many of its details. Since neither Apollonius nor Hipparchus knew the Ptolemaic models there is no doubt that the following theory must be one of Ptolemy's own contributions.

The main idea is to approximate each particular configuration of the Ptolemaic model by a suitable version of the Apollonian model. If, for instance, we wish to examine the retrograde motion of a planet in the neighbourhood of a point with the mean centrum c_m we shall, for this purpose, and at this point only, replace the Ptolemaic

model with an Apollonian model in which the concentric deferent has the radius $TC = \rho(c_m)$ given by (9.22). We shall call this the o s c u l a t i n g A p o l l o n i a n m o d e l corresponding to c_m.

It is obvious that this approximation will be all the more acceptable 1) the smaller the eccentricity is, and 2) the shorter the retrograde period (see page 344) is compared with a complete revolution of the planet. Both conditions express essentially the same thing, viz. that the distance $\rho(c_m)$ of the epicycle centre from the Earth does not vary greatly during one period of retrogradation. Since there is roughly one such period a year, the approximation will be better the longer the period of a complete revolution is.

The stationary points must now be determined by applying (11.2) to the osculating model. Differentiating (9.25) with respect to time we find the angular velocity of the epicycle centre

$$\dot{\lambda}_c = \lambda_m(t) + \dot{q} \tag{11.25}$$

or by (9.12)

$$\dot{\lambda}_c = \omega_t + \dot{q} \tag{11.26}$$

In a similar way the angular velocity of the planet on the epicycle is derived from (9.34)

$$\dot{a}_v = \dot{a}_m(t) - \dot{q} \tag{11.27}$$

or, by (9.13)

$$\dot{a}_v = \omega_a - \dot{q} \tag{11.28}$$

Then (11.2) can be written in the form

$$\frac{\frac{1}{2}SV}{TS} = \frac{\omega_t + \dot{q}}{\omega_a - \dot{q}} \tag{11.29}$$

which is the required generalization of Apollonius' theorem. The equivalent Ptolemaic procedure is found in the chapter on the retrograde motion of Saturn [XII, 2; Hei 2, 471], but in a different form which we can reconstruct in the following way. Two things must here be remembered.

First, we notice that it is no problem to find the derivatives $\lambda_m(t) = \omega_t$ and $a_m(t) = \omega_a$; since the functions $\dot{\lambda}_m$ and a_m are linear functions of time their derivatives are the well known mean motions in longitude and anomaly (argument).

Second, the problem of finding q is quite different because the eccentric anomaly $q(c_m)$ is not an explicit (let alone a linear) function of time, but of the mean centrum c_m as seen from (9.23). Differentiating with respect to c_m we should write

$$\frac{dq}{dt} = \frac{dq}{dc_m} \cdot \frac{dc_m}{dt}$$

or, by (9.15b)

$$\frac{dq}{dt} = \frac{dq}{dc_m} \cdot \omega_t \tag{11.30}$$

Substituting this for q in (11.29) we obtain after reduction

$$\frac{\frac{1}{2}SV}{TS} = \frac{1 + \dfrac{dq}{dc_m}}{\dfrac{\omega_a}{\omega_t} - \dfrac{dq}{dc_m}} \tag{11.31}$$

As we are going to see (page 348), this relation corresponds exactly to the procedure given in the Almagest in the form of a verbal statement, in which dq/dc_m is described as the [increase of] *the equation of the centre corresponding to* [an increase of] *1° in longitude*, which is the same as an increase of 1° in the mean centrum c_m. Accordingly dq/dc_m can be found from the tables of equations as explained below.

One can only guess at the way in which Ptolemy arrived at the idea expressed in (11.30), but perhaps he reasoned as follows: Since the mean centrum c_m is a linear function of time, or, in words more congenial to ancient mathematics, since c_m increases with time in a uniform manner, it can be used as a measure of time. Consequently the rate of change of q with respect to time is the same as its rate of change with respect to c_m – apart from a constant factor expressing the rate of change of c_m itself with respect to time, that is the factor ω_t. Of course, this is no more than a conjecture. But what is indubitably true is that the beautiful generalization (11.31) of Apollonius' theorem is Ptolemy's own work, and a brilliant example of how his mathematical ingenuity and intuitive insight enabled him to solve a most difficult problem.

The General Theory of Retrograde Motions

We are now prepared to tackle the general theory of retrograde motions. Here it will be convenient to introduce a precise notation. The positions of the two stationary points S_1 and S_2 on the epicycle are best defined by their true anomalies

$$a_v(S_1) = \text{arc } A_vS_1 = s_1 \tag{11.32 a}$$

$$a_v(S_2) = \text{arc } A_vS_1S_2 = s_2 \tag{11.32 b}$$

Since S_1 and S_2 are symmetrical with respect to the line TC (see Figure 11.2) we have

$$a_v(S_1) + a_v(S_2) = 360° \tag{11.33}$$

The retrograde arc on the epicycle is

$$\Sigma = S_1\Pi_vS_2 \tag{11.34}$$

and can be expressed

$$\Sigma = a_v(S_2) - a_v(S_1) = s_2 - s_1 \tag{11.34 a}$$

or

$$\Sigma = 2 \cdot (180 - a_v(S_1)) = 2 \cdot (180^\circ - s_1) \tag{11.34 b}$$

The retrograde arc on the ecliptic is the difference

$$\sigma = \lambda(S_2) - \lambda(S_1) \tag{11.35}$$

between the longitudes of S_2 and S_1. This is to be so understood that $\lambda(S_1)$ and $\lambda(S_2)$ must be taken at the times $t(S_1)$ and $t(S_2)$ when the planet arrives at the first and second station respectively. The period of retrogradation is the time interval

$$J = t(S_2) - t(S_1) \tag{11.36}$$

expressed by

$$J = \frac{\Sigma}{\omega_a} \tag{11.37}$$

If Σ is found, J can be determined by the tables of mean motion in anomaly (argument).

The problem is to determine $s_1(c_m)$, $\Sigma(c_m)$ and $J(c_m)$ for a given superior planet corresponding to a given position of the epicycle characterized by the mean centrum c_m. Ptolemy attacks this problem in his usual way, solving it completely for three standard positions, and developing an interpolation method for intermediate positions[5]). The three standard positions are

1) the so-called mean position Q (defined on page 292) at which the epicycle centre has the distance $TC = \rho(c_m{}^\circ) = R$ from the Earth, and the mean centrum $c_m{}^\circ$ is

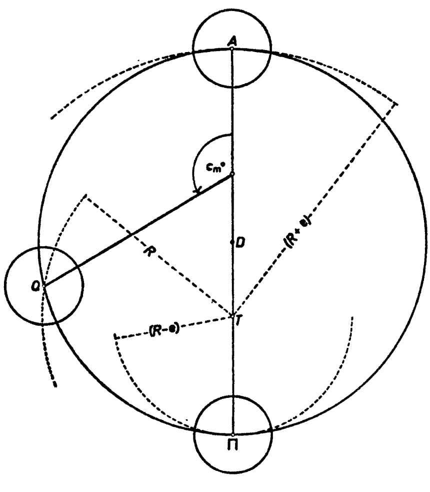

Fig. 11.6

5) Considering this I am unable to understand why Delambre (1817, II, 384) wrote that *Les Grecs ne trouvaient ces points* [...] *que par tatonnement.*

determined by (9.48). At this place we shall denote the values of the three retrogradation functions by Σ°, σ°, and J°.

2) A position near to the apogee A of the deferent defined in such a way that the opposition occurring between S_1 and S_2 takes place exactly at the apogee. Here the values to be found are denoted by Σ', σ', and J'.

3) A position in which the opposition in between S_1 and S_2 takes place exactly at the perigee Π of the deferent. The corresponding values are Σ'', σ'', and J''.

Figure 11.6 shows approximately the three standard positions with the deferent of the corresponding osculating models drawn as dotted circles. In all the three cases the general procedure is the same: Ptolemy derives two equations from which TS_1 and S_1V_1 can be found. Then the function $s_1 = a_v(S_1)$ is determined by simple trigonometrical calculations, and Σ, σ, and J are computed. As in Chapter 9 we shall use the theory of Saturn to illustrate the numerical calculations [XII, 2; Hei, 464].

Retrograde Motions at the Mean Position

Here we have $TC = \rho(c_m{}^\circ) = R$ and consequently

$$TA_v = R + r \quad \text{and} \quad T\Pi_v = R - r$$

Now the first of the two required equations follows by Euclid II, 35 which gives

$$TS_1 \cdot TV_1 = (R + r)(R - r)$$

or

$$TS_1 \cdot (TS_1 + S_1V_1) = (R + r)(R - r) \tag{11.38}$$

With the parameters of Saturn (see page 286) this gives

$$TS_1 \cdot (TS_1 + S_1V_1) = 3557;45 \tag{11.38 a}$$

The second equation is (11.31) adapted to the mean position. According to the small table on page 293, $c_m{}^\circ$ is only slightly different from the value $c_m = 90^\circ$ corresponding to the maximum equation of centre. Consequently we have approximately

$$\frac{dq}{dc_m} = 0 \text{ for } c_m = c_m{}^\circ$$

This reduces (11.31) to the original form

$$\frac{\frac{1}{2}S_1V_1}{TS_1} = \frac{\omega_t}{\omega_a} \tag{11.39}$$

or, in the case of Saturn,

$$\frac{\frac{1}{2}S_1V_1}{TS_1} = 28;25,46 \tag{11.39 a}$$

Now (11.38 a) and (11.39 a) are solved by

$$\tfrac{1}{2}S_1V_1 = 2^p;1,40$$
$$TS_1 \quad = 57^p;38,55$$

whence

$$T\Theta = TS_1 + \tfrac{1}{2}S_1V_1 = 59^p;40,35$$

where Θ is the middle point of S_1V_1 (see Figure 11.7).

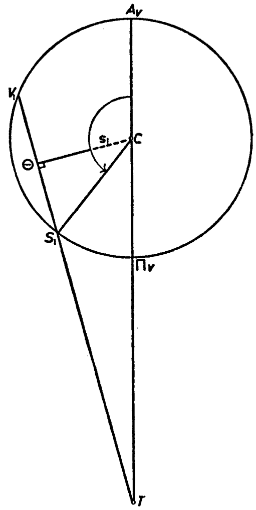

Fig. 11.7

By simple trigonometrical calculations we now find

$$\text{angle } S_1C\Theta = 18°;10,38$$
$$\text{angle } TC\Theta \;= 84°;\ 2,50$$

the difference of which is the supplement to the true anomaly of S_1

$$180° - s°_1 = 180° - a_v°(S_1) = 65°;52,12$$

whence

$$s°_1 = a_v°(S_1) = 114°;\ 7,48$$

and by (11.33)

$$s°_2 = a_v°(S_2) = 245°;52,12$$

The retrograde arc on the epicycle is found from (11.34 a) to be

$$\Sigma° = 131°;44,24$$

and the period of retrogradation from (11.37) to be

$$J° = 138^d$$

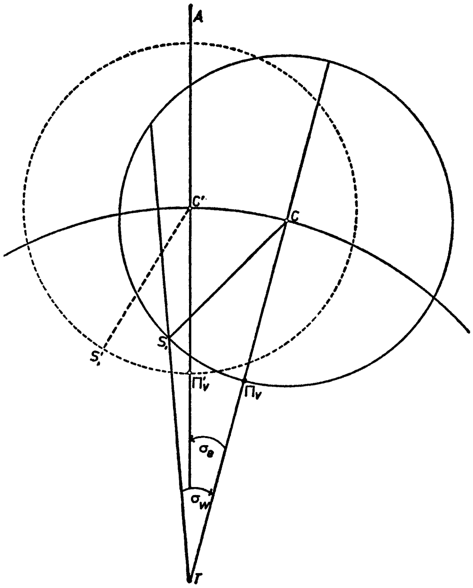

Fig. 11.8

To calculate the retrograde arc on the ecliptic we proceed in the following way (Figure 11.8): If the epicycle centre C were stationary then the motion of the planet P from S_1 to Π_v would cause the line of sight TP to turn westwards through an angle σ_w here given by (cf. Figure 11.7)

$$\sigma_w{}^\circ = \text{angle } S_1TC = 90^\circ - \text{angle } TC\Theta = 5;57,10$$

But this retrograde motion is partially counteracted by an eastward movement σ_e of the line of sight, equal to the motion of the epicycle centre during the time in which P goes from S_1 to Π_v. This angle is determined by

$$\frac{180^\circ - a_v{}^\circ(S_1)}{\omega_a} = \frac{\sigma_e}{\omega_t} \tag{11.40}$$

which here leads to the value

$$\sigma_e{}^\circ = 2^\circ;19$$

Because of the symmetry of S_1 and S_2 the total retrograde arc on the ecliptic will be

$$\sigma = 2(\sigma_w - \sigma_e) \tag{11.41}$$

or, with the values found above,

$$\sigma^\circ = 7^\circ;16,20$$

Thus the problem is solved for the mean position of the epicycle.

The Situation at the Apogee and Perigee

As mentioned above, the second standard position is defined in such a way that the opposition between S_1 and S_2 occurs exactly at the apogee A of the deferent. This implies that when the planet arrives at S_1 the epicycle centre has not yet reached A. Consequently this position is characterized by a mean centrum c'_m a little different from 0°. Therefore the distance $TC = \rho(c_m')$ is not exactly equal to $(R + e)$, where $e = 3^p;25$ is Saturn's eccentricity (see page 282). However, we shall neglect this small discrepancy (as in Figure 11.6) and replace (11.38) by the equation

$$TS_1 \cdot (TS_1 + S_1V_1) = (R + e + r)(R + e - r) \tag{11.42}$$

or in our special case

$$TS_1 \cdot (TS_1 + S_1V_1) = 3979;25,25 \tag{11.42 a}$$

To apply (11.31) to the second standard position considered here we must determine the value of dq/dc_m in the neighbourhood of the apogee of the deferent. The general table of equations of Saturn [XI, 11; Hei 2, 436] shows that

$$q(0°) = 0°$$
$$q(6°) = -0°39$$

whence

$$\frac{dq}{dc_m} \approx \frac{q(6°) - q(0°)}{6} = 0°;6,30$$

Since the value of ω_t/ω_a is the same as in (11.39 a) we can write (11.31) in the form

$$\frac{\frac{1}{2}S_1V_1}{TS_1} = \frac{1 - 0°;6,30}{28;25,46 + 0°;6,30} \tag{11.43}$$

This agrees with Ptolemy's statement that at the first stationary point *the apparent velocity of the epicycle is to the apparent velocity of the planet as 0;53,30 is to 28;32,16* [XIII, 2; Hei 2, 468].

As before we can now solve (11.42 a) and (11.43), finding

$$\tfrac{1}{2}S_1V_1 = 1^p;54,44$$
$$TS_1 = 61^p;11,52$$

Proceeding exactly as in the former case, we arrive at the following results

$$s'_1 = a'_v(S_1) = 112°;44,43$$
$$\Sigma' = 134°;30,34$$
$$J' = 140^d16^h$$
$$\sigma' = 7°;\ 4,20$$

At the third standard position near the perigee the equation (11.42) must be changed to

$$TS_1 \cdot (TS_1 + S_1V_1) = (R - e + r)(R - e - r) \tag{11.44}$$

In (11.31) we have to use the changed value

$$\frac{dq}{dc_m} = \frac{q(183°) - q(180°)}{3} = +0°;7,20$$

Otherwise everything is as in the previous standard position at the apogee. Performing the calculations Ptolemy finds

$$\begin{aligned} s''_1 = a''_v(S_1) &= 115°;28,50 \\ \Sigma'' &= 129°;\ 2,20 \\ J'' &= 136^d \\ \sigma'' &= 7°;18,10 \end{aligned}$$

The Table of Stationary Points

To compare the results of the three sets of calculations we arrange them in the following table, in which some of the values are rounded off.

Position	$s_1 = a_v(S_1)$	Σ	J	σ
First	112°;45	134°;31	141^d	7°; 4
Mean	114°; 8	131°;44	138^d	7°;16
Second	115°;29	129°; 2	136^d	7°;18

This gives some information about the four quantities s_1, Σ, J, and σ conceived as functions of the mean centrum.

It appears that when the epicycle centre moves from the apogee to the perigee the retrograde arc Σ on the epicycle will decrease; on the other hand the retrograde arc σ on the ecliptic will increase at the same time as the period of retrogradation decreases. The problem is to account for these relationships in a more precise and numerical way.

For this purpose Ptolemy calculates for each planet a table with two columns, giving the true arguments $s_1(c_m) = a_v(S_1)$ and $s_2(c_m) = a_v(S_a)$ of the first and second stations as functions of the mean centrum c_m with intervals of 6° from 0° to 180°, that is at equidistant positions of the epicycle centre from the apogee to the perigee of the deferent [XII, 8; Hei 2, 506]. Strictly speaking, none of the tabulated values is known from the outset, first because the mean centrum of the mean position $c_m°$ is not a multiple of 6° (see page 293), and second because we have not yet calculated $s_1(0°)$ or $s_1(180°)$ since the apogee (or perigee) coincides with an opposition but not with a stationary point. Consequently we must begin by investigating the distance of the epicycle centre from the apogee and perigee when the planet is stationary near to these points.

The easiest way of doing this (instead of following Ptolemy's detailed geometrical

calculation in each individual case) is to find the increase Δc_m of the mean centrum c_m during the interval of time in which the planet moves from the first station to the opposition. In other words, we only have to enter the tables of mean motion with half the period of retrogradation J/2 at the apogee (or perigee). This gives the following values, which compare well with Ptolemy's results:

	J/2	$\Delta(c_m)$	Ptolemy
Saturn	$70\frac{1}{3}^d$	2°;21,19	2°;21,18
Jupiter..........	$61\frac{1}{2}^d$	5°; 6,43	5°; 6,35
Mars	40^d	20°;57,44	20°;58,21

Ptolemy maintains [XII, 7; Hei 2, 496] that in the cases of Saturn and Jupiter Δc_m is so small that the value of $s_1(c_m)$ at the exact apogee can be identified with the value at the first station calculated above (page 346). The same situation obtains at the perigee. Consequently we have three values of s_1 for Saturn

$$\left.\begin{array}{l} s_1(0°) \;\;= 112°;45 \\ s_1(c_m°) \;= 114°;\ 8 \end{array}\right\} 1°;23$$
$$s_1(180°) = 115°;29$$

Intermediate values are found by interpolation. Let us consider first the interval

$$0° \leqslant c_m \leqslant c_m°$$

in which the function $s_1(c_m)$ increases by the amount $114°;8 - 112°;45 = 1°;23$ when c_m goes from 0° to $c_m°$, or, as Ptolemy puts it, decreases by 1°;23 when c_m goes from $c_m°$ to 0°. Ptolemy now assumes without proof that the variation of $s_1(c_m)$ depends on the variation of the distance $\rho(c_m)$ of the epicycle centre of the Earth (cf. 9.22). This is evident from the only numerical example illustrating the method [XII, 7; Hei 2, 503], in which he calculates $s_1(30°)$. We have

$$\left.\begin{array}{l} \rho(0°) \;\;= 63^p;25 \\ \left.\begin{array}{l} \rho(30°) = 63^p;20 \\ \rho(c_m°) = 60^p;\ 0 \end{array}\right\} 3^p;20 \end{array}\right\} 3^p;25$$

and calculate

$$s_1(30°) = 114°;8 - 1°;23 \cdot \frac{3;20}{3;25} = 112°;54$$

This procedure corresponds to the general formula

$$s_1(c_m) = s_1(c_m°) - \{s_1(c_m°) - s_1(0°)\} \cdot F(c_m) \tag{11.45}$$

where the interpolation function is determined by

$$F(c_m) = \frac{\rho(0°) - \rho(c_m)}{\rho(0°) - \rho(c_m°)} \tag{11.46}$$

Here Ptolemy has worked backwards from $c_m°$ to a smaller value of the mean centrum. If we wish to start from the lower end $c_m = 0°$ of the interval we can write (11.45) in the form

$$s_1(c_m) = s_1(0°) + \{s_1(c_m°) - s_1(0°)\} \cdot f'(c_m) \tag{11.47}$$

where

$$f'(c_m) = \frac{\rho(c_m) - \rho(c_m°)}{\rho(0°) - \rho(c_m°)} \tag{11.48}$$

Ptolemy does not describe the procedure if the mean centrum lies in the interval

$$c_m° \leqslant c_m \leqslant 180°$$

but (11.47) and (11.48) can easily be adapted to this case, in which we have

$$s_1(c_m) = s_1(c_m°) + \{s_1(180°) - s_1(c_m°)\} \cdot f''(c_m) \tag{11.49}$$

where

$$f''(c_m) = \frac{\rho(c_m°) - \rho(c_m)}{\rho(c_m°) - \rho(180°)} \tag{11.50}$$

These formulae enable us to construct the table of stationary points mentioned above, s_2 following from s_1, by (11.33). Neither the retrograde arcs Σ and σ, nor the retrograde period J are tabulated, but have to be calculated from s_1 and s_2 by the appropriate formula given above.

For Mars the distance of the mean centrum from the apogee at the first stationary point was 20°;58 (page 350), which is too much to allow us to identify $s_1(0°)$ with the value $a_v(s_1)$ as for Saturn and Jupiter. For Mars Ptolemy computes [XII, 7; Hei 2, 497]

	$\rho(0°) = 66^p;0$
$s_1(20°;58) = 157°;47$	$\rho(20°;58) = 65^p;40$
$s_1(c_m°) = 163°;09$	$\rho(c_m°) = 60^p;0$

Postulating the same interpolation procedure as before, these data enable us to calculate $s_1(0°) = 157°;28$ from (11.47) and (11.48), or from (11.45) and (11.46) as Ptolemy does. In the cases of Venus and Mercury we have to proceed in a similar way. With the latter planet there are further complications arising from the alleged existence of its two perigees (page 314); for the sake of brevity we shall not here go into this question, which is dealt with in very much the same way as the question of the equation of argument of Mercury (page 327, cf. Boelk 1911, p. 33 ff.).

The Maximum Elongations of Venus and Mercury

In addition to the retrograde movements one more synodic phenomenon is dealt with in Book XII of the Almagest, viz. the maximum elongations of Venus and

Mercury from the Sun, both towards the West when they appear as morning stars, and towards the East when they are seen as evening stars [XII, 9; Hei 2, 508]. Ptolemy defines the maximum elongation as

$$\eta(\lambda) = \max \{\lambda(t) - \lambda_{\odot}(t)\} \tag{11.51}$$

where $\lambda(t)$ is the true longitude of the planet and $\lambda_{\odot}(t)$ the true longitude of the Sun at the same time t. First, let us consider the case of Venus.

Combining (11.51) and (5.12) we have

$$\eta(\lambda) = \max \{\lambda(t) - \lambda_{m\odot}(t) - q_{\odot}(t)\} \tag{11.52}$$

where $\lambda_{m\odot}(t)$ is the longitude of the Mean Sun at the time of maximum elongation, and $q_{\odot}(t)$ the equation of centre of the Sun at the same time.

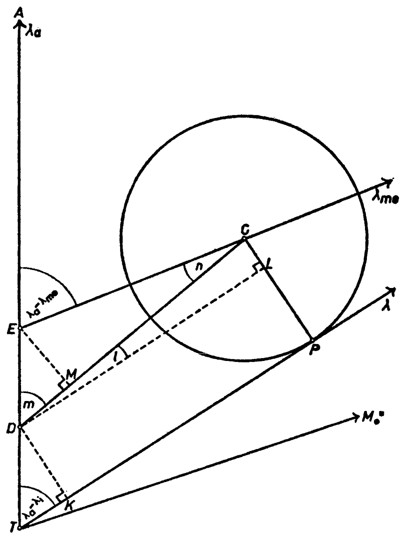

Fig. 11.9

The solution of the maximum problem (11.52) is rather tortuous. First, Ptolemy places the planet P at the beginning of one of the signs on the ecliptic so that its longitude is given as $\lambda_i = i \cdot 30°$ where $i = 0, 1, 2 \ldots 11$. Next, he assumes that this is a position of maximum elongation from the Mean Sun $M_{\odot}{}^*$ so that the line TP (see Figure 11.9) is tangent to the epicycle. Consequently the position of the epicycle centre C is determined which enables us to find the direction $TM_{\odot}{}^* \neq EC$ to the Mean Sun. This position of the Mean Sun determines the date and time t_i at which the planet has the given longitude λ_i. At this time the true Sun has a well defined position in its orbit wherefore also the Sun's equation of centre $q_{\odot}(t_i)$ is defined. He then asserts that the solution to (11.52) is

$$\eta(\lambda_i) = \{\lambda_i - \lambda_{m\odot}(t_i)\} - q_{\odot}(t_i) \tag{11.53}$$

where the last term $q_{\odot}(t_i)$ can be found by the theory of the Sun while the term $\{\lambda_i - \lambda_{m\odot}(t_i)\}$ is determined by the following trigonometrical calculation.

In Figure 11.9 the position of the Mean Sun at the time t_i is determined by the

$$\text{angle CEA} = \lambda_a - \lambda_{m\odot} \tag{11.54}$$

where λ_a is the longitude of the apogee of the planet. A line through D parallel to TP intersects CP in L. DK is perpendicular to TP, and EM perpendicular to DC. We denote the angles CDL by l, EDC by m, and ECD by n. Then we have

$$\text{angle DTP} = \lambda_a - \lambda_i = m + l \tag{11.55}$$

In the triangle TDK we have TD = e, and consequently

$$KD = e \cdot \sin(\lambda_a - \lambda_i)$$

But KD = PL, whence

$$LC = PC - PL = PC - KD = r - e \cdot \sin(\lambda_a - \lambda_i) \tag{11.56}$$

Now the angle l is found from the triangle CDL by

$$\sin l = \frac{LC}{DC} = \frac{r - e \cdot \sin(\lambda_a - \lambda_i)}{R} \tag{11.57}$$

The angle m is found from (11.55) as

$$m = (\lambda_a - \lambda_i) - l \tag{11.58}$$

Next we consider the triangle DEM in which we have DE = e and

$$DM = e \cdot \cos m$$
$$ME = e \cdot \sin m$$

Since the eccentricity of Venus $e = 1^p;15$ is small, the angle n will also be small, so that we have approximately

$$EC \approx MC = DC - DM = R - e \cos m$$

Finally n is found from the triangle ECM by

$$\sin n = \frac{ME}{EC} = \frac{e \cdot \sin m}{R - e \cos m} \tag{11.59}$$

Applying Euclid I, 32 to triangle CED we find the

$$\text{angle CEA} = \lambda_a - \lambda_{m\odot} = m + n$$

or

$$\lambda_{m\odot} = \lambda_a - (m + n) \tag{11.60}$$

where m and n are determined by (11.58) and (11.59) respectively. Thus the maximum elongation of the planet from the Mean Sun is

$$\lambda_i - \lambda_a + (m + n)$$

and from the true Sun, according to (11.52)

$$\eta(\lambda_i) = \lambda_i - \lambda_a + (m + n) - q_\odot(t_i) \tag{11.61}$$

Essentially the same method can be applied to the determination of the maximum elongations of Mercury, with a number of minor changes necessitated by the peculiar geometry of the Mercury model [XII, 9; Hei 2, 514].

The results are given in the form of two short tables [XII, 10; Hei 2, 522] for Venus and Mercury respectively. Each table has 3 columns of 12 rows. Column I contains the names of 12 zodiacal signs from Aries to Pisces. Column II lists the maximum western and Column III the maximum eastern elongations of the planet at its entrance into each sign, that is for $\lambda(t_i) = 0°, 30°, 60° \ldots 330°$.

Final Remarks

Book XII of the Almagest is different from the preceding Books. It is not concerned with constructing planetary theories, but with describing synodic events of interest to traditional, pre-Ptolemaic astronomy. With regard to retrograde movements Book XII revealed that the Ptolemaic theory of longitude was unable to describe this phenomenon in a straightforward way, so Ptolemy had fall back upon the old theorem of Apollonius. From one point of view this was a failure. From another it was a fortunate circumstance which enabled Ptolemy to demonstrate his mathematical ingenuity in a manner rarely surpassed elsewhere in the Almagest, first and foremost through his brilliant extension of Apollonius' theorem (page 342) and through his skilful use of circular inversion. Compared with such feats his theory of maximum elongations seems more pedestrian and devoid of any essential interest. It shows only that this phenomenon could also be mastered and reduced to tabular form.

It would have been natural to include in Book XII a theory of the visibility periods of the planets as a third example of the venerated synodic events. However, since visibility phenomena depend on the latitudes of the planets they cannot be described until we have a theory of latitudes. Such a theory is developed in Book XIII and is the subject matter of the following chapter.

CHAPTER 12

Latitudes and Visibility Periods

Introduction

In all the planetary models of the Almagest the whole machinery of equant, deferent, epicycle, etc., has hitherto been regarded as situated in the plane of the ecliptic. This proved justifiable in so far as the longitudes derived from these models were considered to be in sufficient agreement with experience. Even the complicated longitude variations around the retrograde periods could be accounted for by such flat models as we saw in the preceding chapter. However, until now the various planetary theories have given us an insufficient description of the motions of the planets in so far as even crude observations show that almost all the time they are found outside the ecliptic. In the case of the Moon this is obvious since otherwise we would have a total solar and a total lunar eclipse every month. In order to avoid such a flagrant contradiction with experience it was necessary to develop a theory of the Moon's latitude (see page 200). For the sake of completeness Ptolemy now develops in the final Book of his great composition a similar theory of planetary latitudes. Half of Book XIII (Chapters 1–6) is devoted to this purpose while the remaining chapters (7–10), apart from the final one, give a theory of visibility periods and heliacal risings and settings of the planets.

Following his usual manner of exposition Ptolemy begins with the phenomena of the motion in latitude, but without quoting observations in any detail [XIII, 1; Hei 2, 524]. This survey of the phenomena is followed immediately by a brief outline of the 'hypotheses' i.e. the geometrical models devised to account for them [XIII, 2; Hei 2, 529]. This outline is very abstract and given in a highly condensed form. Moreover, the subject matter itself is far from simple. Therefore, it is understandable that here Ptolemy [XIII, 2; Hei 2, 532] finds time for a digression on the use of the notion of simplicity in astronomy (see above page 28). Here one must remember, says Ptolemy, that it is difficult for man to understand the whole celestial machinery. This is obvious when we think of the labour and difficulty of constructing apparatus simulating the heavenly motions[1]. But to God the heavens may well be simple and easy to understand.

1) Manitius (German transl. vol. 2, p. 334) talks about *Himmelsgloben*, but what Ptolemy had in mind was probably some kind of orrery or planetarium. Already Plato refers to such an instrument (*Timaios* 40 d 2). Later, Archimedes constructed a mechanical model of the universe, driven by water and showing the motion of the Sun, Moon, and planets; Cicero mentions having seen it with his own eyes. Also Posidonius is credited with the construction of a similar apparatus. Hero's model of the universe has been studied by A. G. Drachmann (1971). Finally, in the *Planetary Hypotheses* Ptolemy himself dealt with the construction of a model illustrating the celestial spheres (page 397).

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_12,

This warning against too much anthropomorphism in science leaves Ptolemy free to construct a theory of latitudes just as complicated as he finds it necessary. For complicated it was, and it has been a commonplace in the history of astronomy to comment on its many intricacies. Thus Dreyer wrote that *In no other part of planetary theory did the fundamental error of the Ptolemaic system cause so much difficulty as in accounting for the latitudes* (1906, p. 200), and Delambre confessed that *Tout cela est loin d'être simple et clairement exposé* with the additional remark that *Ptolémé aurait pu s'épargner ce galimatias, et se contenter de mettre sa théorie en Tables* (1817, II p. 395[2]). The latter statement is of course profoundly unjust, since Ptolemy was unable to construct tables except with a geometrical model as basis. Everything and every author considered, it is doubtful whether the theory of latitude has ever been correctly presented as it appears in the Almagest[3]).

One of the reasons why the latitude theories of the Almagest have to be complicated is easily understood from the Copernican point of view. Here the Sun is the central body of the planetary system, also in the sense that the orbit of any planet lies in a plane through the centre of the Sun (or very nearly), just as the orbit of the Moon lies on a plane through the centre of the Earth in both the Ptolemaic and the Copernican system. Therefore it is almost as simple to calculate the heliocentric latitude of the planet according to Copernicus as it was to compute the geocentric latitude of the Moon according to Ptolemy (page 301). But to transform the heliocentric latitude to a geocentric system of reference is no easy task in the Copernican system. To calculate geocentric latitudes directly would be even more difficult – and this is what Ptolemy strives to do.

In the present chapter we shall tentatively try to explain the theory of latitude as it is found in the Almagest, with reference to what Ptolemy regarded as the observational facts upon which it was founded. One of the difficulties is that here even more than elsewhere these facts are described in terms of the finished model. Another is that the fundamental observations are reported in a more cavalier way than in the theory of longitudes. For pedagogical reasons we shall begin with the superior planets.

The Position of the Deferent

Ptolemy begins by explaining that the latitude of a planet is to be considered as an 'anomaly', i.e. a deviation from the smooth circular motion along the ecliptic. This anomaly in latitude has at least two components, connected with the motion in longitude (described by the deferent) and the motion in anomaly (described by the epicycle) respectively. This is stated without further proof, but is easily seen to be true

2) In return Herz (1887, p. 87) speaks of *die von Delambre völlig unverstandene Theorie der Breitenbewegung des Almagests.*
3) The only possible exception is the exposition given by Herz (1887, pp. 143–159). A brief exposition of the essential features of the theory, valid only for the inferior planets, was given by E. S. Kennedy (1951).

without complicated observations. During one complete revolution through the zodiac there is a slow variation in the latitude of a superior planet so that roughly it is northern during one half of the circuit, and southern during the other. But besides this slow variation there is a faster change in latitude in the periods around the retrograde motions connected with the revolution of the planet on the epicycle as seen in Figure 9.1.

In order to account for the change in latitude during a complete revolution along the ecliptic it is necessary to let at least some elements of the geometrical model be situated outside the plane of the ecliptic. There is no doubt that here Ptolemy found inspiration in the theory of the latitude of the Moon. The variation of the Moon's latitude could be accounted for by the assumption that both the deferent and the epicycle are situated in the same plane inclined 5° relative to the ecliptic. This idea seems to have originated with Hipparchus (page 200) whom Ptolemy usually follows as far as possible. But Hipparchus had no planetary theory – as far as we know – so Ptolemy was left to his own resources, and we have no reason to doubt that the theory of planetary latitudes is his own achievement no less than the theory of longitude.

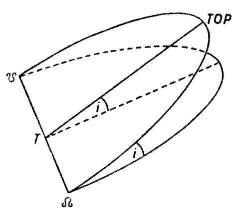

Fig. 12.1

However, the analogy with the theory of the Moon's latitude is manifest from the fact that Ptolemy begins by placing the eccentric deferent in a plane with a certain inclination to the plane of the ecliptic, and passing through the centre T of the Earth, as shown in Figure 12.1. The two planes intersect along a nodal line through T. Following the circumference of the deferent from West to East we find the ascending node ☊ at one end of the nodal line where we pass from southern to northern latitudes. Opposite to ☊ is the descending node ☋ where the deferent passes from northern to southern latitudes. Denoting the ecliptic longitude of ☊ with λ_n, the longitude of ☋ is $\lambda_n + 180°$.

The point of the deferent deviating most from the ecliptic to the North is called by Ptolemy the n o r t h e r n l i m i t of the deferent; there is a corresponding s o u t h e r n l i m i t at the diametrically opposed point. In the following we shall call these points the t o p and the b o t t o m of the deferent respectively. Their ecliptic longitudes will be $\lambda_n + 90°$ for the top and $\lambda_n + 270°$ for the bottom. When dealing with the theory of latitudes Ptolemy usually reckons longitudes from the top of the deferent.

All this is exactly as in the lunar theory (page 200 f.). But there is one important difference. According to (6.68) the nodal line of the Moon rotates in the retrograde direction around the centre of the Earth, while the nodal lines of the planets are supposed

to occupy fixed positions in the plane of the ecliptic relative to the apsidal lines. This, by the way, is the only point where the theory of latitudes is simpler in the case of the five planets than in that of the Moon. The reason for this difference is that the Moon can have its maximum altitude for any longitude, while each of the planets seems to reach its maximum latitude (in the absolute sense) at a definite longitude which can be found from observations.

No details of such observations are given; Ptolemy simply states the ecliptic longitude of the northern limit (or top of the deferent) for each planet. The values are listed in the following table, together with the longitude λ_n of the ascending node and the longitude λ_a of the apogee [cf. XIII, 1; Hei 2, 526].

	Ascending node λ_n	Top of the deferent $\lambda_n + 90°$	Apogee λ_a
Saturn	$\approx 90°$	$\approx 180°$	223°
Jupiter..........	$\approx 90°$	$\approx 180°$	161°
Mars	$\approx 30°$	$\approx 120°$	115°;30

In all three cases we have $0 < \lambda_a - \lambda_n < 180°$. Thus all the superior planets have apogees North of the ecliptic.

The inclination i of the plane of the deferent is now defined as the angle at T between the plane of the ecliptic and a line from T to the top of the deferent [XIII, 1; Hei 2, 525; cf. XIII, 2; Hei 2, 529]. In the case of the superior planets this angle is supposed to be constant. Then the inclined plane is completely determined by i and λ_n.

As for the deferent itself, it is not definitely located in the inclined plane before we have determined the position of its centre D. This is done from the empirical fact that when a superior planet is on the apogeum side of the deferent then its maximum latitude is northern, but southern when it is on the perigeum side [XIII, 1; Hei 2, 525]. This implies that the centre D lies in the inclined plane to the North of the eclip-

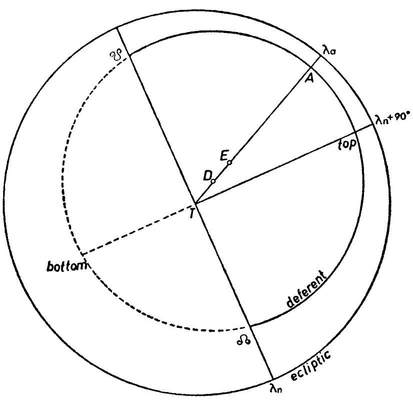

Fig. 12.2

tic. In other words, the nodal line does not divide the deferent symmetrically, but into a northern and a southern part of unequal size, in such a way that the top of the former has a greater distance from the Earth than the bottom of the latter. The exact position of D is determined by the line from the centre T of the Earth to the apogee A of the deferent. Figure 12.2 shows the asymmetrical situation as seen from the North, the dotted lines being underneath the plane of the ecliptic.

The Argument of Latitude

We must now consider the position of the epicycle centre C on the inclined deferent. Here Ptolemy assumes that the ecliptic longitude λ_c of C determined by (9.25) is still valid. This implies that the position of C on the deferent is calculated by means of the 'flat' model used in the theory of longitudes, all the circles of which were situated in the plane of the ecliptic. As in the lunar theory (page 199) this approximation is justified by the small values of the inclination derived below (page 364). The position of C can be defined more conveniently by one or the other of the two following coordinates.

First, we define a variable coordinate by

$$\lambda_d = \lambda_c - \lambda_n \tag{12.1}$$

It measures the distance of C from the ascending node, reckoned from West to East. In the following we shall refer to it as the nodal argument of latitude. It is not much used in the Almagest, where Ptolemy prefers the coordinate

$$\lambda_\beta = \lambda_c - \lambda_n - 90° = \lambda_d - 90° \tag{12.2}$$

It measures the distance of C from the top of the deferent and is called the argument of latitude pure and simple.

The Deviation of the Epicycle

As we have seen (page 200), a similar model with only one inclined circle worked very well in the theory of latitude of the Moon. For the superior planets an equally simple model is impossible because of the complicated latitude variations around the retrograde periods. This second anomaly in latitude must be connected with the epicycle. We know that the centre C of the latter moves upon the circumference of the deferent with an uneven velocity seen from the Earth. Now it is clear that the epicycle cannot lie in the plane of the deferent, for in that case this plane would contain all the lines of sight TP from the Earth to the planet, and the motion of the latter on the epicycle would be unable to influence the latitude determined by the position of C upon the deferent. The plane of the epicycle must have, therefore, a deviation of its own relative to the plane of the deferent.

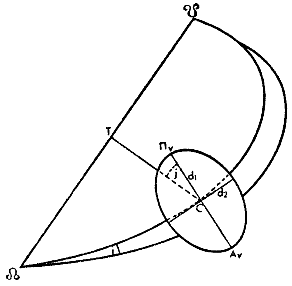

Fig. 12.3

To define the actual position of the epicycle Ptolemy considers first the diameter $\Pi_v CA_v$ containing the true perigee Π_v and the true apogee A_v of the epicycle (see Figure 12.3). This diameter is the line of intersection between the epicycle and a plane through T and C perpendicular to the plane of the ecliptic. We shall (following the Mediaeval Arab astronomers[4]) call this the first diameter of the epicycle and denote it by d_1. The diameter perpendicular to d_1 is called the second diameter of the epicycle and is denoted by d_2. Ptolemy has no particular technical terms for d_1 and d_2.

By means of d_1 and d_2 we can now describe Ptolemy's hypothesis of the motion of the plane of the epicycle in the following way: In the case of the superior planets the second diameter d_2 remains always parallel to the plane of the ecliptic. It serves as an axis about which the first diameter d_1 performs an oscillatory movement. This oscillation of d_1 is coupled to the motion of the epicycle centre along the deferent in such a way that the two periods are the same. The initial condition is such that when C is at the ascending node ☊, d_1 is supposed to lie in the plane of the ecliptic (along the nodal line). When C moves on to the northern latitudes d_1 is tilted about d_2 so that the perigee Π_v moves towards the North, forming a variable angle j called the deviation with the plane of the deferent. When C is at the top of the latter, j has its maximum value j_m. It then begins to decrease, being zero at the descending node, and negative South of the ecliptic when it reaches its minimum value at the bottom of the deferent. Generally speaking d_1 is tilted about d_2 in such a way that the perigee Π_v moves in the same direction (North or South) as the epicycle centre C.

The reason for introducing the deviation of the epicycle is said to be some (unspecified) observations showing that, for a given position of C, the latitude is greater when the planet is observed at the perigee of the epicycle than at the apogee [XIII, 1; Hei 2, 526]. We remember that when the planet is at the perigee it is in opposition to the Sun (page 283). A look at Figure 9.1 shows that here the latitude is, in fact, greater than elsewhere. But when the planet is at the apogee it is in conjunction and consequently invisible. Therefore Ptolemy's statement concerning the latitude at this point cannot derive from direct observations. How he dealt with this difficulty is explained below (page 364).

4) See e.g. al-Biruni, *The Book of Instruction in the Elements of Astrology*, transl. R. R. Wright, London, 1934, p. 102.

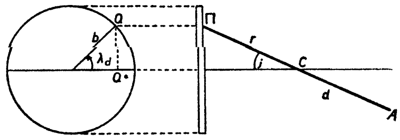

Fig. 12.4

Ptolemy defines the exact form of the d e v i a t i o n function $j(\lambda_d)$ or $j(\lambda_\beta)$ by means of a curious mechanical device shown in Figure 12.4. It comprises a small wheel or circle placed perpendicularly to the plane of the deferent at the perigee end of d_1. This wheel rotates around the line TC with the constant angular velocity ω_t equal to the mean motion in longitude (9.7). When TC points towards the ascending node ☊ the point Q on the circumference of the wheel lies in the plane of the deferent. During the revolution of the wheel Q will cause the true perigee Π_v of the epicycle to move up and down from one side of the plane of the deferent to the other by means of some mechanical link. Figure 12.4 shows the device seen endwise and sidewise in an arbitrary position. If the radius of the wheel is b then we have

$$QQ^* = b \cdot \sin \lambda_d$$

and

$$\sin j = \frac{QQ^*}{r} = \frac{b}{r} \cdot \sin \lambda_d$$

If the deviation can be supposed to be a small angle we have approximately

$$j = j_m \cdot \sin \lambda_d \tag{12.3}$$

where

$$j_m = \frac{b}{r} \tag{12.4}$$

is the maximum deviation. Thus a mechanical device of this kind serves simply to produce an approximately harmonic motion, in this case a harmonic oscillation of the first diameter of the epicycle.

The Values of i and j_m

It follows from the preceding account that the fundamental parameters of the theory of latitude are the longitude λ_n of the ascending node (determined above), the inclination i, and the maximum deviation j_m. In order to find i and j_m Ptolemy has to use empirical data derived from a number of observations. Contrary to the observations upon which the theories of longitude were based, Ptolemy's latitude observations are generally speaking reported in a very unsatisfactory way, neither the dates nor their particular features being indicated. This could raise the suspicion that such

observations were not made by himself, but found in earlier records. Thus he says that in the case of Mars he has chosen two observations of oppositions:

1) one with the epicycle centre at the top of the deferent in which case the latitude was $\beta_1 = 4\frac{1}{3}°$ northern, and

2) another with the epicycle centre at the bottom of the deferent ($\lambda_d = 270°$) where the latitude was $\beta_2 = 7°$ southern [XIII, 3; Hei 2, 539].

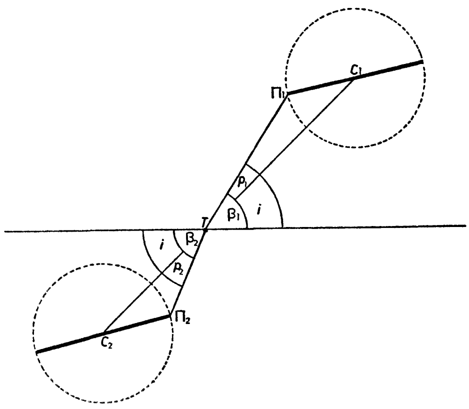

Fig. 12.5

The situation corresponding to these two observations is shown in Figure 12.5, which shows a cross section of the model perpendicular to the nodal line through T. The two oppositions took place at the two positions Π_1 and Π_2 of the true perigee of the epicycle. In the notation of the figure we have

$$i + p_1 = \beta_1 = 4\tfrac{1}{3}° \tag{12.5}$$

$$i + p_2 = \beta_2 = 7° \tag{12.6}$$

from which follows

$$p_2 - p_1 = \beta_2 - \beta_1 = 2\tfrac{2}{3}° \tag{12.7}$$

The angles p_1 and p_2 are not known; but if the epicycles are drawn in the plane of the paper (as the dotted circles in the figure) it becomes clear that these angles are equal to the prosthaphairesis angles corresponding to a true anomaly of $a_v = 180° - j_m$ at the apogee and perigee of the deferent respectively.

Accordingly we have the equations

$$p_1 = p(180° - j_m, 0°) \tag{12.8}$$

$$p_2 = p(180° - j_m, 180°) \tag{12.9}$$

If only j_m were known then each of these angles could be found from the prosthaphairesis table of Mars [IX, 11; Hei 2, 440]. Since j_m is one of the unknown quantities he

seeks to determine, this simple way is closed to Ptolemy. Instead he proceeds in another way which is a new testimony to his ability to handle functions. It appears that he is sufficiently familiar with the prosthaphairesis tables to realize that for small values of j_m the ratio p_1/p_2 will be constant, that is to say independent of j_m. That this is true is easily seen from (9.40a) written in the form

$$\tan p_1 = \frac{r \sin j_m}{R + e - r \cos j_m}$$

$$\tan p_2 = \frac{r \sin j_m}{R - e - r \cos j_m}$$

from which follows

$$\lim \frac{p_1}{p_2} = \lim \frac{\tan p_1}{\tan p_2} = \frac{R - e - r}{R + e - r} \text{ for } j_m \to 0° \qquad (12.10)$$

With the parameters $R = 60;0$, $e = 6;0$ and $r = 39°;30$ of the Mars theory this gives

$$\lim \frac{p_1}{p_2} = \frac{29}{53} = 0.548$$

Now Ptolemy was unable to solve the problem in this formalized manner, so very probably he reasoned as follows: If we put $j_m = 6°$ we find from the prosthaphairesis table that

$$p_1 = 11°;19 - 2;°27 = 8°;52$$
$$p_2 = 11°;19 + 4°;26 = 15°;45$$

with the ratio

$$\frac{p_1}{p_2} = \frac{106}{189} = 0.562$$

Similarly $j_m = 3°$ gives

$$p_1 = 5°;45 - 1°;16 = 4°;29$$
$$p_2 = 5°;45 + 2°;20 = 8°;05$$

and the ratio

$$\frac{p_1}{p_2} = \frac{54}{97} = 0.557$$

This is only a conjecture, for without going into any details at all Ptolemy simply states that

$$\frac{p_1}{p_2} = \tfrac{5}{9} = 0.555 \qquad (12.11)$$

This is a good approximation which together with (12.7) solves the problem with

$$p_1 = 3\tfrac{1}{3}^\circ \quad \text{and} \quad p_2 = 6^\circ$$

By (12.5) or (12.6) we obtain the inclination

$$i = 1^\circ \tag{12.12}$$

With the value $p_1 = 3\frac{1}{3}^\circ$ (12.8) can be written as

$$3\tfrac{1}{3}^\circ = p(180^\circ - j_m, 0^\circ) \tag{12.13}$$

which is solved by the prosthaphairesis table with the result

$$j_m = 2^\circ;15 \tag{12.14}$$

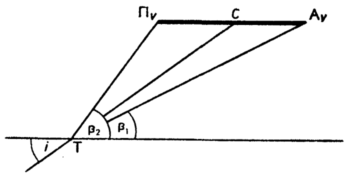

Fig. 12.6

For Saturn and Jupiter it was necessary to use a slightly different method since Ptolemy was unable to find any appreciable difference between latitudes of oppositions at the top and the bottom of the deferent [XIII, 3; Hei 2, 540]. Instead he measured the latitudes of the planets at the apogee A_v and perigee Π_v of the epicycle, i.e. the angles β_2 and β_1 in the preceding Figure 12.6. Since β_2 corresponds to a perigee it can be measured at an opposition as before. On the other hand β_1 corresponds to an apogee of the epicycle and should, therefore, be observed at a conjunction, where the planet is invisible. Here Ptolemy has to content himself with taking the latitude β_1 at the heliacal rising of the planet some time after the conjunction (cf. Figure 9.1). In this way he found $\beta_1 = 2^\circ$ and $\beta_2 = 3^\circ$ for Saturn and $\beta_1 = 1^\circ$ and $\beta_2 = 2^\circ$ for Jupiter. A similar computation as for Mars then gives the following round values

	i	j_m
Saturn	$2\frac{1}{2}^\circ$	$4\frac{1}{2}^\circ$
Jupiter..........	$1\frac{1}{2}^\circ$	$2\frac{1}{2}^\circ$
Mars	1°	$2\frac{1}{4}^\circ$

We notice that the maximum deviation is different from the inclination for all the superior planets[5]). This leads to the conclusion that the plane of the epicycle cannot

5) Thus Dreyer (1906, 198) was wrong when he asserted that *the epicycles in their turn were inclined at the same angles to the plane of the deferents, so that their planes were always parallel to that of the ecliptic.* Later (p. 199) he seems to consider the values of j_m quoted above as foreign to the Almagest and resulting from Ptolemy's changing his mind before he composed the Canopus Inscription.

be parallel to that of the ecliptic except at the nodes, where the two planes are parallel per definitionem. Now we remember that the motion on the epicycle reflects the motion of the Earth around the Sun in the Copernican theory. Therefore the planes of all the three epicycles ought to be parallel to the ecliptic if the two systems were equivalent. Consequently we must conclude that the Ptolemaic theory of latitude cannot be transposed to the Copernican system without a change of parameters.

The General Latitude Function

Having determined the fundamental geometric parameters of the latitude theories of the three superior planets, we must now develop a general procedure for calculating the latitude of any of these planets at an arbitrary point of its course. It would seem that the latitude is a function of three variables, viz. the position of the epicycle centre expressed by its ecliptic longitude λ_c, the tilting of the epicycle expressed by the deviation angle j, and the position of the planet on the epicycle expressed by the true argument a_v. But since the variation of j is coupled to λ_d, which again depends only on λ_c according to (12.1), there really are but two degrees of freedom; the latitude will accordingly be a function of two variables, viz. a_v and either λ_c or λ_d. In the following we find it most convenient to determine it as a function $\beta = \beta(\lambda_d, a_v)$ of λ_d and a_v, while Ptolemy prefers λ_β and a_v. This will be put right in the final programme (page 368).

Ptolemy attacks the problem in a way which is characteristic for many of his computations in planetary theory: First he selects two standard positions of the epicycle centre, viz. the top and the bottom of the deferent. Next he shows how the latitude corresponding to any given value of the anomaly a_v can be found at these positions, by means of a numerical example illustrating the general procedure of that part of the calculation. Finally he proposes an approximation method for finding the latitude at places outside the standard positions. The results of his own calculations are tabulated in such a way that a latitude can be found by a few simple arithmetical operations involving no trigonometric computations. Here we shall analyse his procedure as it appears in a special example of how the latitude of Saturn is found at a position where the epicycle centre is at the top of the deferent (i.e. where $\lambda_d = 90°$ and $\lambda_\beta = 0°$) and the planet has a true anomaly of 135° [XIII, 4; Hei 2, 553].

Figure 12.7 illustrates this situation. A plane is placed through T perpendicular to the nodal line. It intersects the deferent in C and the plane of the epicycle in d_1. The planet P is projected upon the plane of the ecliptic in Q and upon d_1 in K. The latter point is projected upon the plane of the ecliptic in B and upon the line TC in M. The latitude to be found is the angle

$$\beta(90°, a_v) = \text{angle QTP}$$

In triangle CPK the angle at C is $(a_v - 180°)$ and the angle at K is 90°. This gives

$$CK = r \cos a_v \tag{12.15}$$

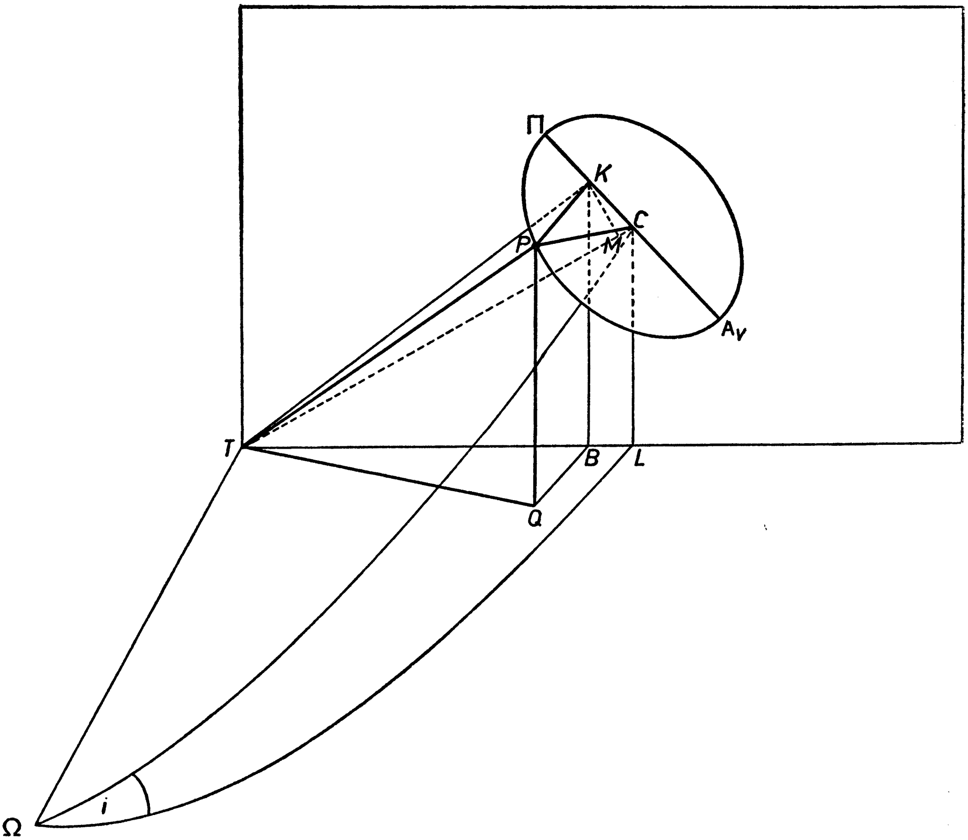

Fig. 12.7

$$KP = r \sin a_v \tag{12.16}$$

In triangle CKM the angle at M is 90° and the angle at C is j_m, whence

$$MK = CK \sin j_m = r \sin j_m \cdot \cos a_v \tag{12.17}$$

$$CM = CK \cos j_m = r \cos j_m \cdot \cos a_v \tag{12.18}$$

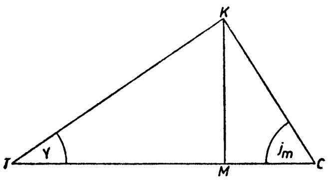

Fig. 12.8

Next we consider triangle TCK (see Figure 12.8), in which TC = $\rho(c_m)$ can be computed by (9.22), c_m being the mean centrum at the top of the deferent, where the true centrum is

$$c = 90° + \lambda_n - \lambda_a$$

Then by (9.24)

$$c_m = 90° + \lambda_n - \lambda_a - q(c_m) \tag{12.19}$$

The solution of this transcendental equation (similar to Kepler's equation) must be found numerically by the table of the equation of the centre. We shall denote the corresponding value of $\rho(c_m)$ by the constant ρ'.

Since the angle at T is small, we have approximately

$$TK \approx TM = TC - CM = \rho' - r \cos j_m \cdot \sin a_v \tag{12.20}$$

The angle CTK $= \gamma$ can be found from

$$\sin \gamma = \frac{MK}{TK} = \frac{r \sin j_m \cdot \cos a_v}{\rho' - r \cos j_m \cdot \sin a_v} \tag{12.21}$$

Since

$$i + \gamma = \text{angle BTK}$$

we have

$$QP = BK = TK \cdot \sin (i + \gamma) \tag{12.22}$$

In TPK the hypotenuse is

$$D = TP = \sqrt{TK^2 + KP^2} = [(\rho' - r \cos j_m \sin a_v)^2 + (r \sin a_v)^2]^{\frac{1}{2}} \tag{12.23}$$

cf. the analogous formula (9.39).

Then the latitude to be found is determined by

$$\sin \beta(90°, a_v) = \frac{QP}{TP} \tag{12.24}$$

A more formal expression derived from the results above is

$$\sin \beta(90°, a_v) = \frac{(\rho' - r \cos j_m \sin a_v) \cdot \sin (i + \gamma)}{D(\rho', j_m, a_v)} \tag{12.25}$$

where γ is determined by (12.21) and D by (12.23).

For this position of the epicycle centre, β will be a function $\beta(a_v)$of the true anomaly only. Thus it is possible to compute $\beta(a_v)$ for regular intervals of anomaly, and to tabulate the results for a_v and $360° - a_v$ in Column 3 of the latitude table for Saturn [XIII, 5; Hei 2, 582].

A completely similar calculation is now carried out for position in which the epicycle centre is at the bottom of the deferent, i.e. for $\lambda_d = 270°$; and the results are tabulated in Column 4 of the same table.

So much for the two standard positions. Regarding the latitude at an arbitrary position of the epicycle as a function $\beta = (\lambda_d, a_v)$ of both the argument of latitude λ_d and the true anomaly a_v, we are now faced with the problem of determining this

general function from the two known (tabulated) functions $\beta(90°, a_v)$ and $\beta(270°, a_v)$. Without any proof and from a rather vague analogy to the formula (6.64) for the latitude of the Moon Ptolemy postulates that all we have to do is to reduce the function $(90°, a_v)$ with the factor $\sin \lambda_d$, if we are above the ecliptic on the top part of the deferent of $(0° \leqslant \lambda_d \leqslant 180°)$. Consequently the latitude is here expressed by

$$\beta(\lambda_d, a_v) = \sin \lambda_d \cdot \beta(90°, a_v) \tag{12.26}$$

On the bottom part of the deferent $(180° \leqslant \lambda_d \leqslant 360°)$ we have the corresponding formula

$$\beta(\lambda_d, a_v) = \sin \lambda_d \cdot \beta(270°, a_v) \tag{12.27}$$

Ptolemy therefore adds a 5th Column to the table of latitudes, containing the function $\sin \lambda_d$, expressed as a sexagesimal fraction and computed as

$$\sin \lambda_d = \frac{\beta_{☽}(\lambda_d)}{5} = 0;12 \cdot \beta_{☽}(\lambda_d) \tag{12.28}$$

where $\beta_{☽}$ is the latitude of the Moon tabulated in the 7th Column of the general prosthaphairesis table of the Moon [V, 8; Hei 1, 390].

The general programme for calculating the latitude of a superior planet at a given time t can be summarized as follows [XIII, 6; Hei 2, 587].

1) Find the longitude λ_c of the epicycle centre by formula (9.25) of the theory of longitudes
2) Find the true argument from (9.34).
3) Find the argument of latitude λ_β from (12.2) and also λ_d from (12.1).
4) Enter into the latitude table with λ_β and take the number in Column V.
5) Enter into the latitude table with a_v and take the number in Column III if $0° \leqslant \lambda_d \leqslant 180°$, or in Column IV if $180° \leqslant \lambda_d \leqslant 360°$.
6) Multiply the two numbers obtained from the table.

It is characteristic of the method that Ptolemy determines the latitude of a superior planet directly by (12.24), without distinguishing between that part of β which is due to the inclination of the deferent and that stemming from the deviation of the epicycle[6]). This is explicitly underlined by Ptolemy himself [XIII, 4; Hei 2, 543]. Later we shall see how this procedure is modified in the theory of the inferior planets.

The Inferior Planets

In the case of Venus and Mercury the theory of latitude differs from that of the superior planets in several ways. In general it is more complicated. This is to be ex-

6) In the literature on the Ptolemaic theory of latitude it is a common mistake to assume that the latitude function of the superior planets is a sum of 2 components, stemming from the inclination and the deviation respectively. This error may be due to the fact that Ptolemy based his *Handy Tables* on a theory of this kind in order to establish a uniform procedure for calculating latitudes of all the five planets (cf. below page 400).

pected, since we here have larger epicycles so that the motion of the planet on the epicycle will influence the latitude to a greater extent. In the following we shall describe the underlying geometrical model of the theory in general terms before proceeding to the derivation of parameters and the working out of a general latitude function. The real or alleged observational data upon which the theory is founded will be mentioned below, where we are going to determine the numerical parameters of the model (page 371).

Ptolemy tries to account for the latitudes of Venus and Mercury by a kinematical model which is most easily explained as the model for the superior planets described above with these essential modifications:

1) First, for both Venus and Mercury the apsidal line of the deferent is supposed to be perpendicular to the nodal line. This means that

$$\lambda_a = \lambda_n + 90° \tag{12.29}$$

In other words, the top of the deferent is at the apogee. At this point both the true and the mean centrum are zero. Using (12.2) we find

$$\lambda_\beta = \lambda_c - \lambda_a = c \tag{12.30}$$

Accordingly the argument of latitude of an inferior planet is equal to its true centrum. The nodal argument of latitude (12.1) is

$$\lambda_d = c + 90° \tag{12.31}$$

This feature represents a simplification while the following makes the model rather more complicated.

2) As with the superior planets, the plane of the deferent of Venus or Mercury is inclined at an angle i relative to the plane of the ecliptic. However, the inclination is no longer constant, but an oscillating function giving the plane of the deferent a rocking motion about the nodal line [XIII, 1; Hei 2, 528]. This motion is coupled with the movement of the epicycle centre C on the deferent in such a way that the inclination is zero when C is at the nodes, but maximum when C is at the perigee A or the apogee Π. In a way this oscillation of the deferent is analogous to the deviation of the epicycle of a superior planet and has the same period. Thus when the epicycle centre of Venus moves from the ascending node $\lambda_d = 0°$ the apogee will rise from the plane of the ecliptic to a maximum latitude at $\lambda_d = 90°$, only to sink back again at the descending node $\lambda_d = 180°$. When C continues towards the perigee this point will now rise above the ecliptic, reaching a maximum at $\lambda_d = 270°$. This coupling means, accordingly, that the epicycle centre of Venus is never below the ecliptic. For Mercury the opposite is true, and here C is never above the ecliptic [XIII, 2; Hei 2, 529].

If we define the inclination as the angle between $\overrightarrow{TA}$ and the plane of the ecliptic, we can express it by

$$i = i_m \cdot \sin \lambda_d \tag{12.32}$$

where the maximum value i_m is positive in the case of Venus, but negative for Mercury.

3) In the cases of the superior planets the first diameter d_1 of the epicycle had a variable angle of deviation j relative to the plane of the deferent coupled with the motion of the epicycle centre C and determined by (12.3). For Venus and Mercury there is a similar deviation of d_1. But the phase is different, the maximum of j being at the nodes, while j is zero at the apogee or perigee. Thus we may replace (12.3) by

$$\sin j = \sin j_m \cdot \sin(\lambda_d + 90°) \tag{12.33 a}$$

or, approximately,

$$j = j_m \cdot \sin(\lambda_d + 90°) \tag{12.33 b}$$

defining j as the angle between the vector $\overrightarrow{C\Pi_v}$ and the plane of the deferent (as in the superior planets). If a northern deviation is reckoned as positive, we have $j_m < 0$ for Venus, and $j_m > 0$ for Mercury.

4) In the cases of the superior planets the second diameter d_2 of the epicycle remained parallel to the plane of the ecliptic. In the cases of the inferior planets it is given a variable slant k, defined as the angle between the plane of the ecliptic and the eastern half of d_2. This gives d_2 an oscillatory motion, which is coupled to the motion of the epicycle C along the deferent in the same way as the deviation of d_1, but again with a different phase [XIII, 2; Hei 2, 530]. The movement is produced by a wheel mechanism analogous to that described above (page 361) and placed at the eastern end of the mean position of d_2. Accordingly the angle of slant can be expressed by

$$k = k_m \cdot \sin \lambda_d \tag{12.34}$$

where $k_m > 0$ for Venus, but < 0 for Mercury.

The General Behaviour of the Model

Since both the inclination i, the deviation j, and the slant k are variable functions of the argument of latitude λ_d, their combined effect is to give the epicycle a heaving, pitching, and rolling motion like that of a ship in a heavy sea. To get an impression of the overall effect we shall consider the changing situations of the epicycle of Venus at four principal points in its course round the deferent.

1) C is at the ascending node, where $\lambda_d = 0°$ and $\lambda_\beta = 270°$. Since $i = 0°$ the plane of the deferent lies in the plane of the ecliptic. The first diameter d_1 has its maximum deviation $j = -j_m$ with the perigee end points upwards relative to the plane of the ecliptic. The latter contains the second diameter d_2 since the slant $k = 0°$ in this position.

2) C arrives at the top of the deferent, that is the apogee A, where $\lambda_d = 90°$ and $\lambda_\beta = 0°$. The plane of the deferent has its maximum inclination $+i_m$, and C its maximum northern distance from the ecliptic. The deviation is $j = 0°$, so that the diameter d_1 is parallel to the plane of the deferent. The slant $k = +k_m$ has its maximum value, so that the eastern end of d_2 is pointing northwards.

3) C reaches the descending node, where $\lambda_d = 180°$ and $\lambda_\beta = 90°$. Here the inclination $i = 0°$ and the plane of the deferent has again come down to that of the

ecliptic. The deviation being $j = +j_m$, the perigee end of the 1st diameter is pointing upwards. Since the slant $k = 0°$, the 2nd diameter lies in the plane of the deferent (ecliptic).

4) C is at the 'bottom' or perigee Π of the deferent, with $\lambda_d = 270°$ and $\lambda_\beta = 180°$. But now this point is lifted above the ecliptic, the inclination having its maximum negative value $i = -i_m$. Thus also C has its maximum northern distance from the ecliptic. The deviation is $j = 0°$ and d_1 lies in the plane of the deferent. The slant is $k = -k_m$ and the eastern end of d_2 is pointing downwards.

So much for Venus. In the case of Mercury the motion of the deferent takes place in the inverse order, so that the epicycle centre will be below the ecliptic both at the 'top' and the bottom of the deferent, and at all intermediate positions except at the nodes. Also the deviation and slant of the epicycle are reversed.

Determination of the Constant i_m

The first step towards a quantitative theory of the latitudes of Venus and Mercury must be the determination of the maximum values of the inclination, deviation, and slant by means of a number of suitably selected observations. To these Ptolemy unfortunately refers in a rather vague manner.

Let us first consider a situation in which the epicycle centre C is at the apogee A (or the perigee Π) of the deferent [XIII, 1; Hei 2, 527; cf. XIII, 3; Hei 2, 535]. We know from the survey above that here the first diameter d_1 (that is, the line $A_v\Pi_v$) lies in the plane of the deferent. It follows that if the planet P is at A_v or Π_v then it has the same latitude as C. Such observations are said to show that at these positions Venus has a northern latitude of 0°;10 and Mercury a southern latitude of 0°;45. Since the latitude of C at the top of the deferent (that is, the apogee) defines the maximum inclination, we have

$$i_m = +0°;10 \text{ for Venus } (+ \text{ meaning northern}) \qquad (12.35\text{ a})$$

$$i_m = -0°;45 \text{ for Mercury } (- \text{ meaning southern}) \qquad (12.35\text{ b})$$

This derivation raises the obvious question of the nature of the observation upon which it is founded. We remember that when an inferior planet is at the apogee or perigee of the epicycle then it is in lower conjunction with the Sun, and therefore invisible. How can its latitude be observed in these situations?

The answer is that such observations are impossible. Contrary to what he asserts in the Almagest, Ptolemy simply cannot have found the two crucial latitudes by direct observation. Does this mean that the observations are faked and that Ptolemy is guilty of fraud? Or that he proceeded in the same way as in the cases of Saturn and Jupiter, where a similar difficulty arose? If the latter be the case one wonders why Ptolemy passed it over in silence.

To shed a little more light on this problem we first ask whether Ptolemy's result is correct. Conjunctions of Venus and the Mean Sun can be determined from the true longitudes of Venus and the Sun taken from Tuckerman's Tables, and the equation

of the Sun taken from the table found in the Almagest [III, 6; Hei 1, 253]. For the apogee of Venus we use the Ptolemaic value $\lambda_a = 55°$ (page 300), and for that of the Sun the value $\lambda_a = 65°;30$ (page 147). Looking for conjunctions in the neighbourhood of $\lambda = 55°$ we find for instance

	A.D. 88	
	May 10	May 20
Longitude of Sun	47°.88	57°.41
Argument of Sun	342°.38	351°.91
Equation of Sun	0°.68	0°.37
Longitude of Mean Sun	47°.20	57°.04
Longitude of Venus	46°.09	58°.37
Latitude of Venus	−0°.38	+0°.01

Using linear interpolation we find that a conjunction of this kind occurred on A.D. 88, May 15, at the longitude $\lambda \approx 52°$, when Venus had a southern latitude $\beta = -0°.19 = -0°;11$.

Now the small eccentricity of the deferent of Venus (page 302) makes the assumed position $\lambda_a = 55°$ rather uncertain. Nevertheless, the longitude $\lambda = 52°$ of the conjunction is sufficiently near to the established apogee to justify a comparison between Ptolemy's alleged observation and the latitude calculated above. The numerical values 0°;10 and 0°;11 agree well enough, but the sign is wrong. Actually Venus had a southern latitude on the date in question, while Ptolemy maintains that it is northern at all such conjunctions.

Our calculations also confirm that Venus was invisible at this particular conjunction. The equation of the Sun on May 15 A.D. 88 was about 0°;53. This is the distance in longitude between Venus and the true Sun. This value combined with the small latitude makes the true angular distance too small for Venus to be seen.

Thus there is little doubt that here Ptolemy started from an erroneous result. Another question is how it was found when direct observation was impossible? Assuming that Venus is visible when its elongation is about 5° from the true Sun (see page 388) we find from Tuckerman's Tables for A.D. 88:

	April 30	June 4
Longitude of Sun	38°.32	71°.69
Longitude of Venus	33°.82	76°.79
Elongation	4°.50	5°.10
Latitude of Venus	−0°.74	+0°.59

Here the mean value of the two latitudes is −0°.8. This is again negative and contradictory to Ptolemy's assumption. Thus we are forced to conclude that how Ptolemy obtained his basic data remains a mystery.

Determination of the Constant k_m

The determination of the maximum slant k_m is a more complicated affair, but being founded on data actually obtainable from real observations, it is also more satisfactory from an astronomical point of view. Again we consider a situation in which the epicycle centre C is at the top (or bottom) of the deferent. Here the slant has the maximum value k_m according to (12.34). The observation is made when the planet has its maximum eastern or western elongations from the Sun. It is said to show that when Venus appears as evening star (eastern elongation) its latitude is about 5° greater than when it appears as morning star (western elongation) [XIII, 3; Hei 2, 535]. Thus the maximum slant of the epicycle makes the latitude of Venus differ by about 2°;30 from the latitude of the epicycle centre. In other words, the maximum slant gives the planet a latitude of 2°;30 relative to the inclined plane of the deferent.

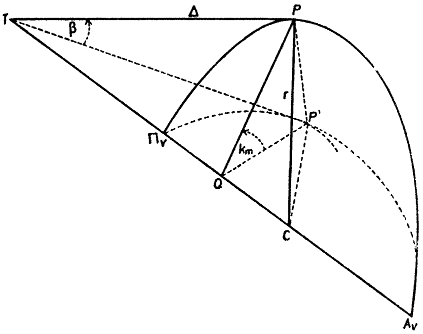

Fig. 12.9

Figure 12.9 shows the epicycle in the position of maximum slant around the first diameter $A_v\Pi_v$, which points towards T. A circle of radius r is drawn around the epicycle centre C in a plane parallel to the plane of the ecliptic. The planet is at P, and TP is tangent to the epicycle because the elongation is maximum, while P'T is tangent to the other circle. It follows that TP = TP' and that from P and P' we can draw perpendiculars to the same point Q on the first diameter. Then angle P'QP = k_m is the maximum value of the slant. In the usual notation

$$TC = \rho, \quad CP = CP' = r \quad \text{and} \quad TP = TP' = \Delta$$

we have from the right angled triangle TP'C

$$\Delta^2 = \rho^2 - r^2$$

and by Euclid VI.4

$$\frac{\Delta}{h} = \frac{\rho}{r}$$

where $h = QP'$. The angle $P'TP = \beta$ is approximately the latitude of the planet relative to the plane of the deferent containing C, or 2°;30 according to the observations quoted above. Ptolemy uses the law of chords (3.7) to find

$$PP' = 2\Delta \cdot \sin \beta/2$$

and also

$$PP' = 2h \cdot \sin k_m/2$$

The three relations combine into

$$\sin k_m/2 = \frac{\rho}{r} \cdot \sin \beta/2 \tag{12.36}$$

For small angles this is approximately the same as

$$\sin k_m = \frac{\rho}{r} \cdot \sin \beta \tag{12.36 a}$$

or

$$k_m = \frac{\rho}{r} \cdot \beta \tag{12.36 b}$$

from which k_m can be found. In his numerical calculation [XIII, 4; Hei 2, 572] Ptolemy uses the values

$$\beta = 2°;30, \quad r = 43^p;10, \quad \rho = 60^p$$

where $\rho = R + e = 61^p;15$ was the correct value at the apogee of the deferent. The result is for Venus

$$k_m = 3°;30 \tag{12.37 a}$$

which is a round value for 3°;28,29. For Mercury the result is

$$k_m = 7°;0 \tag{12.37 b}$$

calculated at the mean distance $\rho = R + e = 63^p;0$ [XIII, 4; Hei 2, 574].

Determination of the Constant j_m

Finally we consider a situation in which the epicycle centre C is at one of the nodes and thus 90° from the apogee. Here the second diameter d_2 lies in the plane of the ecliptic (page 370), while the first diameter has its maximum deviation j_m. Ptolemy maintains as a result of observations that when Venus is at the apogee A_v of the epicycle it has a northern latitude of 1°, but a southern latitude of 6°;20 when it is at the perigee [XIII, 3; Hei 2, 536]. Here we are faced with the same difficulty as before, viz. that the planet is invisible in such positions. Consequently we are here also forced to

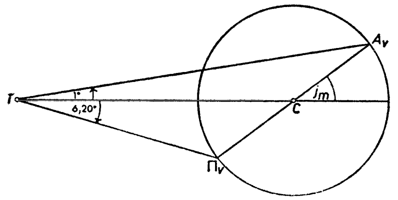

Fig. 12.10

the conclusion that the alleged observational latitudes were provided by methods which Ptolemy did not wish to disclose. Assuming them to be correct, we can derive the maximum deviation j_m in almost the same way as in the case of Saturn (page 362). Thus Figure 12.10 leads to the relations

$$1° = p(j_m, c_m°) \tag{12.38 a}$$

$$6°;20 = p(180° + j_m, c_m°) \tag{12.38 b}$$

where p is the equation of argument of Venus (page 309) and $c_m°$ the mean centrum at the mean position, which is approximately a node. By the table of equations of Venus (Column VI) we find from (12.38 a) $j_m = 2°;23$, and from (12.38 b) $j_m = 2°;32$. Ptolemy proceeds in a slightly different way. Taking

$$j_m = 2°;30 \tag{12.39 a}$$

as a hypothetical value, he uses the table to find the equation of argument; at the apogee the result is 1°;2 and at the perigee 6°;22, which are in sufficient agreement with the alleged observational values to make the hypothetical value acceptable.

In Mercury the 'observations' show that when the epicycle centre is at the ascending node the planet has a southern latitude of 1°;45 at the apogee of the epicycle, but a northern latitude of 4° at the perigee. This leads to

$$j_m = -6°;15 \tag{12.39 b}$$

in the same way as before [XIII, 3; Hei 2, 536].

The Problem of the General Latitude Function

Having found the value of the parameters i_m, j_m and k_m we must now try to determine a function expressing the latitude of the planet at any given position on the epicycle, characterized by the true argument, a_v, and for any given position of the moving, deviating and slanting epicycle relative to the ecliptic. Since both the deviation (12.33) and the slant (12.34) are functions of the same variable λ_d, which also governs the inclination (12.32) of the deferent and expresses the position of the epicycle centre, we realize that a general latitude function must have two independent variables λ_d and a_v just like the latitude function of the superior planets. But while the latter was

expressed directly by (12.25–26), the complicated motion of the epicycles of Venus and Mercury forces Ptolemy to account for their latitudes by an approximative method only.

Actually Ptolemy deals with the problem as if the fact that the planet wanders away from the plane of the ecliptic had three separate and mutually independent causes, viz. the inclination, deviation, and slant. If they really are independent then their combined effect will be the sum of three contributions, each of which can be determined without regard to the others. In other words, Ptolemy tries to write the latitude in the form

$$\beta(\lambda_d, a_v) = \beta_1(\lambda_d) + \beta_2(\lambda_d, av) + \beta_3(\lambda_d, a_v) \tag{12.40}$$

Here the first latitude $\beta_1(\lambda_d)$ is due only to the inclination of the deferent, and is thus independent of a_v. The second latitude $\beta_2(\lambda_d, a_v)$ is an effect of the deviation, and the third latitude $\beta_3(\lambda_d, a_v)$ originates from the slant. This means that because of the inclination the centre C of the epicycle is deviated β_1 degrees above the ecliptic. The deviation lifts the planet β_2 degrees above the level of C, and the slant gives it an extra elevation β_3.

As usual in the Almagest, the final result is summarized in the form of two tables of latitude for Venus and Mercury respectively [XIII, 5; Hei 2, 582]. Each table has 5 columns and 45 rows, and in each table Column I contains the argument of latitude λ_β defined by (12.2) at intervals of 6° from 6° to 90°, and of 3° from 90° to 180°. Column II contains the argument (360° $-$ λ_β). The three other columns must now be examined separately.

The Effect of the Inclination

Let us first consider the effect of the inclination. We know that when the nodal argument of latitude is λ_d the epicycle centre C is situated in a plane forming an angle given by (12.32) with the ecliptic. This means that we can determine the latitude β_1 of C by the simple approximation formula of the lunar theory if we substitute (12.32) for i in (6.55). This gives

$$\beta_1 = \beta_1(\lambda_d) = i_m \cdot \sin^2 \lambda_d \tag{12.41}$$

showing that β_1 has the same sign as i_m, that is positive for Venus and negative for Mercury[7]) according to (12.35 a–b) [XIII, 6; Hei 2, 589]. The function (12.41) is not tabulated. But column V of the latitude tables contains (both for Venus and Mercury) the function sin ($\lambda_d + 90°$) expressed as sexagesimal functions; thus it is identical with the function (12.28) tabulated in Column V in the latitude tables of the superior planets (see page 368). Ptolemy's procedure for calculating β_1 is accordingly,

7) I think this is what is behind the curious statement by Manitius, that *die Bahn der Venus keinen niedersteigenden, die Bahn des Merkurs keinen aufsteigenden Knoten hat* (see his German Almagest-translation, vol. 2, p. 328). What is true is, of course, that the first latitude component (due to the inclination) is always northern in the case of Venus, and southern in the case of Mercury.

first, to find the tabulated sexagesimal fraction of i_m, and next the same fraction of this result, in complete agreement with (12.41).

The Effect of the Deviation

To find the effect of the deviation, Ptolemy first selects a position of the epicycle where both the inclination and the slant are zero [XIII, 4; Hei 2, 546]. This happens at the ascending node where, $\lambda_d = 0°$ and the centre C is near its mean position on the deferent (cf. page 375). The deviation of the first diameter d_1 has its maximum

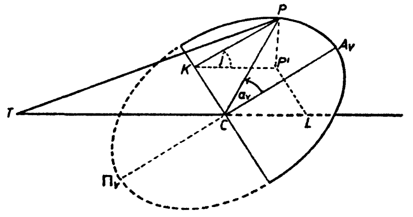

Fig. 12.11

value j_m given by (12.39). In Figure 12.11 the planet P is projected on the plane of the deferent – coinciding with the ecliptic – at P′ and upon the second diameter d_2 at K; P′ is projected on the first diameter at L. The true argument is the angle $A_vCP = a_v$, and the maximum deviation is angle $P'KP = j_m$. The distance TC is denoted by ρ as usual. The latitude to be found is the angle $P'TP = \beta_2(\lambda_d, a_v)$, or $\beta_2(0°, a_v)$ for $\lambda_d = 0°$.

We find from the figure

$$LP' = CK = r \sin a_v$$
$$KP = r \cos a_v$$

and

$$P'P = KP \sin j_m = r \sin j_m \cdot \cos a_v \tag{12.42}$$

$$KP' = KP \cos j_m = r \cos j_m \cdot \cos a_v \tag{12.43}$$

For TL we have

$$TL = TC + CL = TC + KP' = \rho + r \cos j_m \cdot \cos a_v \tag{12.44}$$

and for TP′

$$TP' = \sqrt{TL^2 + LP'^2}$$

We can now determine the latitude by

$$\tan \beta_2(0°, a_v) = \frac{P'P}{TP'}$$

but Ptolemy has to use the equivalent relation

$$\sin \beta_2(0°, a_v) = \frac{P'P}{TP} \tag{12.45}$$

where

$$TP = \sqrt{TP'^2 + P'P^2}$$

Using the expressions above we can write (12.45) in the form

$$\sin \beta_2(0°, a_v) = \frac{r \sin j_m \cdot \cos a_v}{D(\rho, j_m, a_v)} \tag{12.46}$$

where

$$D(\rho, j_m, a_v) = [(\rho + r \cos j_m \cdot \cos a_v)^2 + (r \sin a_v)^2 + \\ + (r \sin j_m \cdot \cos a_v)^2]^{\frac{1}{2}} \tag{12.47}$$

is the distance TP of the planet from the Earth for $\lambda_d = 0$, corresponding to the function (9.39) in the flat models.

These relations reproduce Ptolemy's procedure as explained by him in a numerical example. The function $\beta_2(0°, a_v)$ (12.46) is tabulated in Column III of the latitude tables under the assumption that ρ has its mean value R.

There remains the problem of determining the general function $\beta_2(\lambda_d, a_v)$ for $\lambda_d \neq 0$, that is for positions of the epicycle centre outside the ascending node. Ptolemy postulates [XIII, 6; Hei 2, 588] that this can be done by a procedure equivalent to the formula

$$\beta_2(\lambda_d, a_v) = \sin(\lambda_d + 90°) \cdot \beta_2(0°, a_v) \tag{12.48}$$

where the factor $\sin(\lambda_d + 90°)$ is the function tabulated in Column V. This implies several approximations.

First, in (12.46) we replace $\sin \beta_2$ by β_2. This involves only a small error since the latitude $\beta_2(0°, a_v)$ is a small angle. Combining (12.46) and (12.48) we obtain the latitude

$$\beta_2(\lambda_d, a_v) = \frac{r \sin j_m \cdot \sin(\lambda_d + 90°) \cdot \cos a_v}{D(\rho, j_m, a_v)} \tag{12.49}$$

According to (12.33 a) the equation (12.49) can be written as

$$\beta_2(\lambda_d, a_v) = \frac{r \sin j \cdot \cos a_v}{D(\rho, j_m, a_v)} \tag{12.50}$$

Comparing this expression with (12.46) we see that the numerator in (12.49) would have emerged if we had derived (12.46) from a situation in which the deviation was j instead of j_m. So far, so good. But in the latter position the denominator should also have been adjusted – first, by writing j instead of j_m, and second, by replacing $\rho = R$

with the correct value of the new distance from T to C. Ptolemy disregarded these adjustments, presumably convinced that the small eccentricities and small values of j_m allowed him to regard the denominator as constant.

The Effect of the Slant

Here we begin by placing the epicycle centre C at the top (apogee) of the deferent, where the slant has its maximum value, while the deviation (but not the inclination) is zero. The first problem is to find the third latitude $\beta_3(90°, a_v)$, defined above (page 376) as a function of a_v at this particular position of the epicycle. The second problem is to generalize the resulting expression, ending with β_3 as a general function of λ_d and a_v. More than elsewhere in the theory of latitudes Ptolemy's methods and procedures for reaching this goal are obscure and marked by uncontrolled approximations. Before investigating them we shall therefore start with the derivation of $\beta_3(90°, a_v)$ as a function expressed explicitly in modern trigonometrical terms.

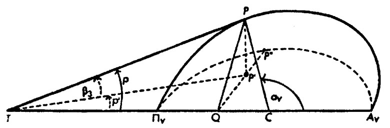

Fig. 12.12

Figure 12.12 shows one half of the epicycle slanting upwards relative to the plane of the deferent. The planet P is projected upon the first diameter d_1 at Q, and upon the plane of the deferent at P′. The dotted circle shows the mean position of the epicycle (without slant) lying in the plane of the deferent and intersected in P″ by QP′ produced. The angle $P'QP = k_m$ is the maximum slant. The latitude angle is $P'TP = \beta_3(90°, a_v)$, and the angle $CTP = p$ is the elongation of the planet from C, that is, from the mean Sun, measured in the plane of the epicycle and equal to the angle CTP″. This elongation is the angle under which the actual radius CP of the epicycle is seen from the Earth, and therefore equal to the equation of argument $p(c_m, a_v)$ determined by (9.40) or (10.37).

Since the latitude we seek is β_3 = angle P′TP, we have

$$P'P = TP \cdot \sin \beta_3 = QP \cdot \sin k_m \tag{12.51}$$

or

$$\sin \beta_3 = \sin k_m \cdot \frac{QP}{TP} = \sin k_m \cdot \sin p \tag{12.52}$$

from which relation we conclude that β_3 is maximum at a position of P at which the equation of argument p is maximum too, that is, at the maximum elongation of P from the mean Sun. Ptolemy arrives at the same conclusion by a geometrical argument

based on inequalities [XIII, 4; Hei 2, 568]. To determine β_3 explicitly as a function of a_v we only have to express QP and TP in (12.52) by a_v. We have

$$QP = r \cdot \sin a_v \tag{12.53}$$

and

$$TP = \Delta(\rho, a_v) = [TP'^2 + P'P^2]^{\frac{1}{2}} = [TQ^2 + QP'^2 + P'P^2]^{\frac{1}{2}} \tag{12.54}$$

With

$$TQ = TC + CQ = \rho + r \cos a_v$$

$$QP' = QP \cdot \cos k_m = r \cos k_m \sin a_v$$

and

$$P'P = QP \cdot \sin k_m = r \sin k_m \sin a_v$$

we can write (12.54) in the form

$$TP = \Delta(\rho, a_v) = [(\rho + r \cos a_v)^2 + (r \sin a_v)^2]^{\frac{1}{2}} \tag{12.55}$$

The latitude is then determined by

$$\sin \beta_3(90°, a_v) = \frac{r \sin k_m \cdot \sin a_v}{\Delta(\rho, a_v)} \tag{12.56}$$

where Δ given by (12.55) is identical with the distance expressed by (9.39). Comparing (12.56) with the expression (9.40) or (10.37) for the equation of argument, we see that (12.52) is satisfied. With the proper value of

$\rho = R + e$ for Venus
$\rho = R + 3e$ for Mercury

(12.56) is the complete solution of the first problem.

Ptolemy's procedure for finding the third latitude $\beta_3(90°, a_v)$ at any point on the epicycle as a function of the true argument a_v is not equivalent to (12.56) but derives from another line of reasoning which, perhaps, we can reconstruct as follows: Starting at the true apogee of the epicycle ($a_v = 0°$) we know that here both β_3 and the equation of argument p are zero. With increasing values of a_v they reach their respective maxima at the same point (of maximum elongation). For a_v increasing towards 180° (the true perigee), they both decrease to zero. Generalizing this crude relation, Ptolemy maintains [XIII, 4; Hei 2, 580] that the ratio of the latitude to the true argument is independent of a_v and can be determined by the relation

$$\frac{\beta_3}{p} = \frac{\max \beta_3}{\max p} \tag{12.57 a}$$

or

$$\beta_3 = \frac{p}{\max p} \cdot \max \beta_3 \tag{12.57 b}$$

As proof of this relation Ptolemy states that the ratio QP:QP′ (see Figure 12.12) is independent of the position of Q on the first diameter [XIII, 4; Hei 2, 570]. From (12.52) it follows that

$$\sin \beta_3 = \frac{\sin p}{\sin \max p} \cdot \sin \max \beta_3 \tag{12.58}$$

Consequently (12.57) is valid only for small values of the latitude β_3 and the argument p. In both Venus and Mercury β_3 amounts at most to a few degrees, while p has a maximum of 46° and 28° degrees respectively. Thus the approximation (12.57) has a rather dubious character and ought to have been more precisely examined. However, omitting this Ptolemy finds it necessary to discuss some of his other approximations before he proceeds to define and tabulate the third latitude in its final form.

The Problem of the Equation of Argument

The function p in the expression (12.57) for β_3 is the angle under which the epicycle radius CP is seen from the Earth T. At the position considered here – the apogee of the deferent – this angle is not affected by the slant of the epicycle, and can be found in the general tables of equations pertaining to the theory of longitudes as the equation of argument $p(c_m, a_v)$, where $c_m = 0°$ according to (12.30). Thus there is no real problem as to what p means. However, Ptolemy is aware that p can be understood as

1) either the equation of argument giving the proper correction of the longitude in the 'flat' model, and also necessary for the calculation of latitudes by (12.57),
2) or an auxiliary function in the theory of latitudes which happens to be tabulated in the theory of longitudes, but which is not the proper correction to be used in the latter.

For, if we adopt the second point of view, it is clear that the difference between the mean and the true longitudes of an inferior planet is the

$$\text{angle } CTP' = p'$$

in Figure 12.12. In other words, the equation of argument, conceived as a correction to the ecliptic longitude of one of these planets, is influenced by the slant of the epicycle to the amount of

$$p - p' = \text{angle } PTP'' \tag{12.59}$$

It follows that we would have to correct the equation of argument found in the tables by this amount were it of sufficient importance to be taken into account. To this question Ptolemy devotes some numerical calculations. But first he proves a theorem stating that not only the proper equation of argument p, but also its projection p′ on

the plane of the deferent is maximal at the greatest elongation [XIII, 4; Hei 2, 569]. By Figure 12.12 we have

$$\sin p' = \frac{QP'}{TP'} = \frac{QP'}{QP} \cdot \frac{QP}{TP} \cdot \frac{TP}{TP'}$$

where

$$\frac{QP'}{QP} = \cos k_m, \quad \frac{QP}{TP} = \sin p, \quad \frac{TP}{TP'} = \frac{1}{\cos \beta_3}$$

whence

$$\sin p' = \frac{\cos k_m}{\cos \beta_3} \cdot \sin p \tag{12.60}$$

Now p has a maximum at the greatest elongation. At the same point β_3 has its maximum value; consequently $\cos \beta_3$ is minimal, and $1/\cos \beta_3$ maximal, and we can conclude that Ptolemy's statement is true.

Ptolemy is less fortunate when he continues with the assertion that also the correction (12.59) to the equation of argument is maximal at the greatest elongation, like p and p′ taken separately. This conclusion is obviously false, and so is his proof, which amounts to showing that the ratio P′P″:TP is greater here (at the greatest elongation) than elsewhere on the epicycle. Now we have (see Figure 12.12) that

$$\begin{aligned} \frac{P'P''}{TP} = \frac{QP'' - QP'}{TP} &= \frac{QP''}{TP} - \frac{QP'}{TP} \\ &= \frac{QP}{TP} - \frac{QP'}{QP} \cdot \frac{QP}{TP} \\ &= \sin p(1 - \cos k_m) \end{aligned} \tag{12.61}$$

which shows that the ratio P′P″: TP in fact has its maximum value where p is maximal, that is at the greatest elongation. But this is no proof that (p − p′) is maximal at the same point, since P′P″ : TP is proportional to sin p, but no correct measure of the difference (p − p′). The latter would be true only if Q, P, P′, and P″ were points on a common sphere with T as its centre. In other words, Ptolemy's argument implies that he has treated the plane triangle QPP″ as if it were a spherical triangle and P′ a point on the arc QP″.

After these theoretical deliberations Ptolemy calculates the numerical values

$$p - p' = 0°;1 \text{ for Venus}$$
$$p - p' = 0°;6 \text{ for Mercury}$$

with the epicycle centre at the mean distance of the deferent [XIII, 4; Hei 2, 574 and 2, 576]. These values are considered small enough to make the influence of the slant on the equation of argument insignificant. We may conclude accordingly that the general tables of equations are suitable and accurate enough to be used both in the theory of longitude as well as in the theory of latitude. The transition from the 'flat' to the inclined model can be made without affecting the tables.

The Varying Distance of the Epicycle Centre

There still remains one more approximation to be investigated. In Column VI of the general table of equations we find the equation of argument $p(a_v, c_m{}^\circ)$ as a function of a_v taken at the mean distance $\rho = R = 60^p$ in the case of Venus, or $\rho = R + e = 63^p$ in the case of Mercury (see page 303 and 316). We also saw above (page 374) that Ptolemy calculated the maximum slant k_m using these values of ρ, notwithstanding the fact that he started with data related to a situation in which the epicycle centre was at the apogee of the deferent, far from the mean distance. The quality of this approximation has to be checked before we can begin to calculate a general table of latitudes.

Beginning with Venus [XIII, 4; Hei 2, 576], Ptolemy makes the assumption that for this purpose the maximum value (12.37a) of k_m (found at the mean distance) is sufficiently accurate. With this value he determines the latitude β_3 at the maximum and minimum distances of the epicycle centre. Then the computed latitudes are compared with the value of β_3 from which k_m was found. A similar method is used for Mercury [XIII, 4; Hei 2, 578]. The following table shows the results

		Mean	Maximum	Minimum
Venus	ρ	$60^p;0$	$61^p;15$	$58^p;45$
	β_3	$2^\circ;30$	$2^\circ;27$	$2^\circ;34$
Mercury	ρ	$63^p;0$	$69^p;0$	$57^p;0$
	β_3	$2^\circ;30$	$2^\circ;17$	$2^\circ;46$

In the case of Venus the maximum variation in β_3 is only $0^\circ;4$, which Ptolemy regards as insignificant. Consequently the table of latitudes can be calculated as if the epicycle centre were always at the mean distance from the Earth, or – which is the same thing – as if Venus had a concentric deferent. For Mercury the variation of β_3 is

$$\begin{aligned} &2^\circ;30 - 2^\circ;17 = 0^\circ;13 \text{ at the apogee} \\ &2^\circ;30 - 2^\circ;46 = -0^\circ;16 \text{ at the perigee} \end{aligned} \tag{12.62}$$

Since this amounts to a fraction equal to $0^\circ;16 : 2^\circ;30 = 11\%$ of the latitude at the mean position, Ptolemy decides that the influence of the variation of ρ had better be taken into account in the latitude theory of Mercury.

The Final Expression of β_3

At last Ptolemy is able to decide upon the final procedure to determine the third latitude β_3 of Venus and Mercury. The basic formula in both cases is (12.57b), but there is a small difference in the way in which it is applied to each planet [XIII, 4; Hei 2, 580].

In the case of Venus the influence of the distance $\rho = TC$ of the epicycle centre on β_3 is considered to be of no importance. Consequently Ptolemy ignores it, using the values max $\beta_3 = 2°;30$ and max $p = 46°$ (page 381) determined above. This gives

$$\beta_3(90°, a_v) = \frac{2;30}{46} \cdot p = 0;3,16 \cdot p(90°, a_v) \qquad (12.63)$$

Taking $p(90°, a_v)$ from Column II of the general table of equations [XI, 11; Hei 2, 442], the function (12.63) is tabulated in Column IV of the latitude table of Venus [XIII, 5; Hei 2, 584].

What remains is to replace (12.63) by a function which also accounts for the influence of the mean longitude on β_3. Now the angle of slant is coupled with the motion in longitude by (12.34). There Ptolemy postulated (without proof) that outside the apogee of the deferent not only the slant, but also the third latitude will be reduced by the factor $\sin \lambda_d$ which is already tabulated in Column V of the latitude table. Considering, first, that in all the relations above the slant always appears as a factor, and second, that the angle is small, this is a reasonable assumption. The result is the final formula

$$\beta_3(\lambda_d, a_v) = \sin \lambda_d \cdot \beta_3(90°, a_v) \qquad (12.64)$$

It follows that for Venus β_3 is found by multiplying two numbers on the same line of Columns IV and V of the latitude table.

In the case of Mercury Ptolemy begins in the same way as in Venus by substituting the Mercury parameters max $\beta_3 = 2°;30$ (page 383) and max $p = 22°$ into (12.57b). This gives [XIII, 4; Hei 2, 580]

$$\beta_3(90°, a_v) = \frac{2°;30}{22} \cdot p(90°, a_v) = 0;6,49 \cdot p(90°, a_v) \qquad (12.65)$$

as the function tabulated in Column IV of the latitude table of Mercury, the equation of argument $p(90°, a_v)$ being taken – as for Venus – from Column VI of the table of equations. But now the procedure has to be modified owing to the influence of the varying distance of the epicycle centre, which is deemed too great to be ignored. Because of (12.62) Ptolemy decides that if we enter one of the first 15 lines of the latitude table then the values in Column IV should be diminished by 10% before being multiplied by the factor $\sin \lambda_d$ taken from Column V. Similarly they should be increased by 10% in the rest of the table [XIII,6; Hei 2, 588]. This means that we have to split the relation corresponding to (12.64) into two formulae, viz.

$$\beta_3(\lambda_d, a_v) = \tfrac{9}{10} \cdot \sin \lambda_d \cdot \beta_3(90°, a_v) \qquad (12.66a)$$

for

$$0° \leqslant \lambda_d \leqslant 180°$$

and

$$\beta_3(\lambda_d, a_v) = \tfrac{11}{10} \cdot \sin \lambda_d \cdot \beta_3(90°, a_v) \tag{12.66b}$$

for

$$180° \leqslant \lambda_d \leqslant 360°$$

It appears that this procedure does not introduce any discontinuity of β_3.

The Final Formula

At last we have come to the end of the theory of latitude of Venus and Mercury. According to (12.40) and (12.41), (12.48) and (12.64–66) the final formula can be written

$$\begin{aligned} \beta(\lambda_d, a_v) = {} & \sin \lambda_d \cdot \beta_1(90°, a_v) \\ & + \sin(\lambda_d + 90°) \cdot \beta_2(0°, a_v) \\ & + \sin \lambda_d \cdot f \cdot \beta_3(90°, a_v) \end{aligned} \tag{12.67}$$

where $f = 1$ in the case of Venus, and 0.9 or 1.1 in the case of Mercury. In the Almagest the three terms are, of course, represented by their numerical values, and combined according to a detailed set of rules [XIII, 6; Hei 2, 588] for deciding whether the resulting latitude is southern or northern. Having given the three terms the signs defined above there is no need for us to go into these rules here. Concerning the practical use of (12.67) it should be remembered that Ptolemy always uses $\lambda_\beta = \lambda_d - 90°$ as argument in the tables of latitude (page 359).

Concluding Remarks on the Theory of Latitudes

Let us here compare the final formulae (12.67) and (12.26–27) for the latitude of an inferior and a superior planet respectively. They reveal that Ptolemy tried to make the procedures as similar as possible, the factor $\sin \lambda_d$ being common to both formulae, and therefore tabulated in the last column of all five latitude tables. On the other hand there is one obvious difference. In (12.67) each term represents the contribution of a single variable to the total latitude, defining the inclination, the deviation and the slant; but in (12.26–27) the second factor (12.25) depends upon both the deviation of the epicycle and the inclination of the deferent. This means that (12.25) is an exact expression without the many approximations used in deriving the various terms in (12.67). That it is nevertheless possible to tabulate (12.25) as a function of one variable (see page 367) is, of course, due to the fact that the superior planets have constant inclinations. Consequently, there is no need to express (12.26–27) as a sum of two different terms, one representing the effect of the inclination, and another that of the deviation. Therefore in the Almagest Ptolemy does not introduce a first and second latitude of the superior planets, contrary to what is often stated in the literature on the subject. It is only in the cases of Venus and Mercury that the total latitude is split

into three components, each representing the effect of a single cause (cf. Note 12.6, page 368 and page 400).

Heliacal Risings and Settings

At last Ptolemy is able to finish his great exposition with a small and marginal problem which could not be dealt with until the theory of latitudes was completed. Considering the great trouble he took to work out this theory it is a curious fact that he applied it only to one single phenomenon, of no fundamental importance to his astronomy taken as a whole.

In a previous chapter (page 265 f.) we have mentioned the heliacal rising and setting of the planets, or their first and last visibilities, as synodic phenomena of great importance to the earlier astronomers. In fact, such events were useful to observers devoid of instruments as a simple means of obtaining empirical data to be used in theoretical astronomy, primarily for calendaric purposes. At the time of the Almagest they had lost their original importance for reasons which we have stated above (page 267), and it is understandable that Ptolemy gives them a rather cavalier treatment, and presumably includes them only out of traditional veneration for a classical problem. Thus he gives a very superficial account of the observations upon which his theory is founded, and also the methods employed are to a large extent clad in obscurity. This makes it difficult to interpret this final part of the Almagest, although a recent paper by A. Aaboe (1960a) has cleared most of the ground.

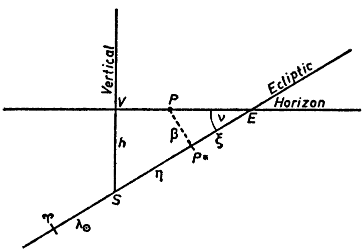

Fig. 12.13

To understand the problem as it is dealt with by Ptolemy we consider Figure 12.13, which shows the ecliptic intersecting the eastern or western horizon in a point E with the ecliptic longitude $\lambda = \lambda(E)$ under the angle v, which according to II,11 (see page 116) depends on $\lambda(E)$ and upon the geographical latitude φ. The Sun S is below the horizon; it has the negative altitude $h = VS$ and the longitude $\lambda_{\odot}$. A star P is seen just above the horizon and projected upon the ecliptic at P*. It has the longitude $\lambda(P)$ and the latitude $\beta(P) = P^*P$, its (eastern or western) elongation from the Sun being

$$\eta = SP^* = \lambda(P) - \lambda_{\odot}$$

The arc P*E is called ζ. We then have

$$\eta + \zeta = \lambda - \lambda_{\odot} \tag{12.68}$$

Considering Figure 12.13 as plane, we find by simple geometrical considerations

$$h = \eta \sin v + \beta \cos v \tag{12.69}$$

upon which relation Ptolemy's theory is founded.

Now suppose that the elongation η is western, and just great enough to make the corresponding value of h so large, that the Sun is sufficiently below the horizon to make the star visible in the early morning just before sunrise. In this case the star rises heliacally, and is again seen after a period of invisibility. The smallest value of h for which this is possible was called by Latin astronomers the *arcus visionis* of the star.

Similarly the figure may refer to a situation in which the star is seen after sunset just above the horizon on the last day of the visibility period. This event is its heliacal setting. We can now distinguish the following three cases.

1) If P is a fixed star it will be overtaken once a year by the eastward moving Sun. Its heliacal setting will then be followed by a period of invisibility before its heliacal rising takes place.

2) If P is any one of the superior planets then it will also have an eastward motion when found in the neighbourhood of the Sun, since retrograde motions occur around oppositions (page 263). But in all three cases this motion is slower than that of the Sun; therefore the latter will also overtake the superior planets once a year, and produce the same phenomena as in the fixed stars.

3) If P is one of the inferior planets then the situation is a little more complicated, partly because of their limited elongations and partly because their mean daily motions are greater than that of the Sun. This means that there will be first and last visibilities both in the East and the West. For instance, if Venus is direct then it will overtake the Sun and – like the Moon – have both a first visibility as an evening star, and a last visibility as a morning star. The opposite happens if it is retrograde and thus overtaken by the Sun.

We can now state the general problem of visibility in the following terms:

> To find the least possible elongation of the star from the Sun compatible with its being visible just above the horizon before sunrise or after sunset.

Ptolemy's solution of this problem is briefly indicated in a single chapter of the Almagest [XIII, 7; Hei 2, 590], in which he is concerned only with the five planets. The solution is founded upon (12.69) written in the form

$$\eta = \frac{h - \beta \cos v}{\sin v} \tag{12.70}$$

Thus the desired arc η will in general depend upon the values of h, β and v.

The Standard Arc of Vision

Among these parameters h will in general depend on the brightness of the star and the atmospheric conditions under which it is observed. Therefore Ptolemy begins by establishing a standard value of h for each of the five planets from empirical data obtained under the best possible conditions. He does not say how this material was found, but he has presumably searched the Alexandrian archives for records of the exact dates of heliacal risings and settings near midsummer. This point of time has, in fact, a double advantage. First, the atmosphere is usually clear enough to make good observations of phenomena on the horizon possible. Second, the Sun is near the beginning of Cancer, and since the point E in Figure 12.13 is near to the Sun we have $\lambda(E) \approx 90°$. This implies that in the neighbourhood of E the ecliptic is parallel to the equator and forms equal angles with the horizon both in the morning in the East and in the evening in the West.

Unfortunately Ptolemy does not reveal how h is calculated from the dates of the observations. The most plausible hypothesis is that he used (12.69), which implies that he was able to determine the values of v, η and β at any given date. In fact, v can be found by the method described in II, 11 (page 116 f.) if only $\lambda(E)$ and the geographical latitude φ are known. The elongation η is the difference between the true longitudes of the planet and the Sun, and can accordingly be calculated by the theories of longitudes described above in Chapter 5 and 9. Finally the latitude of the planet can be found from the theory developed in the first part of the present chapter. But there is no trace of the actual calculations, and Ptolemy restricts himself to quoting the results listed in the following table

Planet	h
Saturn	11°
Jupiter	10°
Mars	$11\frac{1}{2}$°
Venus	5°
Mercury	10°

The values appear to be in qualitative agreement with the relative brightnesses of the planets. They differ from the values used in the *Handy Tables*.

The Visibility Tables

The five values of h being found, the problem of visibility is solved by (12.70), in which the remaining two parameters β and v are functions only of the time and the geographical latitude. But since Ptolemy's mathematical notation does not permit him to cope with a formalized expression of such a function, he has to fall back upon his

usual method of tabulating η for a discrete set of properly chosen values of the independent variables. How this is done is explained in a very inadequate way, which clearly reveals the small importance he attached to the whole question.

First, Ptolemy restricts himself to one single geographical latitude ($\varphi = 33°;18$), defined as the latitude of the 10th parallel passing through Phoenicia, where the longest day is $14\frac{1}{4}^h$ (see page 108). This choice is expressly motivated by the fact that the 10th parallel also passes through the regions where formerly the Chaldaeans made their observations [XIII, 7; Hei 2, 594]. Since the latitude of Babylon is 32°;40, this is nearly true. Ptolemy also gives the reason that the Greeks observed a little to the North and the Alexandrians a little to the South of this parallel. Since according to Ptolemy (page 108) the latitudes of Rhodes and Alexandria are 36°;0 and 30°;22 respectively, the two places are fairly distant from the 10th parallel. This reveals that Ptolemy is more concerned with showing by an example how the problem can be solved theoretically than with providing Hellenistic astronomers with tables for universal use.

The angle v between the ecliptic and the horizon depends not only on φ, but also on the longitude $\lambda(E)$ of the rising or setting point E in Figure 12.13. Here Ptolemy chooses the beginnings of each of the 12 signs; we have accordingly

$$\lambda(E) = 0°, 30°, 60°, \ldots 330°$$

The actual calculation of v for these values of λ is not illustrated in the Almagest. There is no doubt that Ptolemy had to use the method developed in II,11 (page 116).

There remains the problem of how Ptolemy determined the values of the latitude $\beta(t)$ of the various planets to be used in (12.70) at each of the 12 positions of E. We know from the theory presented in the first part of this chapter that $\beta(t)$ can be found if only t is known. But here t is not given immediately – only the longitude of the point E near which the planet is found. That this difficulty may be overcome can be shown in the following way.

From Figure 12.13 we deduce the relation

$$\lambda_{\odot} = \lambda(E) - SE = \lambda(E) - \frac{h}{\sin v} \tag{12.71}$$

from which we can find the true longitude $\lambda_{\odot}$ of the Sun for each position of E, since h is known and v calculated as shown above.

Next we find the longitude λ_a of the apogee of the Sun by (8.12). Subtracting λ_a from $\lambda_{\odot}$ according to (5.18), we get the true argument a of the Sun, which by the table of equations [III, 6; Hei 1, 253] leads to an approximate value of the equation of the centre $q(a_m)$. Subtracting the equation from $\lambda_{\odot}$ according to (5.12), we find the mean longitude $\lambda_{\odot m}(t)$ of the Sun. But this is a linear function of time given by (5.2). Consequently the time t can be found from the tables of the mean motion of the Sun.

What Ptolemy actually did does not appear with complete clarity from the numerical examples he gives. But in the case of Venus and Mercury [XIII, 7; Hei 2, 595] he

explains that he found the mean position of the Sun and calculated the corresponding latitudes of the planets. This indicates that he did, in fact, proceed according to the general trend of the above argument.

The final table [XIII, 10; Hei 2, 606] summarizes his results. For each planet Column I gives the names of the signs. In the superior planets Column II contains the values of η corresponding to the beginning of each sign and to the heliacal rising of the planet in the East, while Column III gives the values of η corresponding to the heliacal setting in the West. For the inferior planets two more columns are necessary, since we must here account for the first and last visibility of each of these planets both as morning and evening stars. A detailed analysis of these tables has been given by Aaboe (1960 a).

Conclusion

Ptolemy devotes a particular chapter [XIII, 8; Hei 2, 597] to a numerical proof of the fact that his invisibility theory is able to describe the very peculiar behaviour of Venus and Mercury. Observations show that when Venus is at the beginning of Pisces only two days (or less) elapse between its last appearance as an evening star and its first appearance as a morning star, while this period is 16 days when Venus is at the beginning of Virgo. Mercury is even more irregular, since it does not appear as an evening star at the beginning of Scorpio, nor as a morning star at the beginning of Taurus. We shall not examine this part of the theory here.

It is a fact worth noticing that Ptolemy disregards the visibility phenomena of the Moon. This is all the more remarkable in view of the enormous role these phenomena played in Babylonian astronomy. In Mesopotamia the first appearance of the crescent of the Moon marked the beginning of a new month, and it was thus an important and difficult problem for the astronomers to develop methods of predicting this event. Of this there is no trace left in the Almagest. In fact it was unnecessary for Ptolemy to give any attention to this problem, since he based all his astronomical work on the Alexandrian calendar with its equal months of 30 days, the beginning of which can be found by simple enumeration.

Here in Book XIII Ptolemy disregards also the theory of heliacal risings and settings of the fixed stars. It was rather briefly dealt with already in a previous chapter [VIII, 6; Hei 2, 198], but without any numerical details whatever. This does not mean that Ptolemy was uninterested in these phenomena which he utilized in his work *Phaseis* for particular calendaric purposes (see Vogt, 1920).

CHAPTER 13

Epilogue – the Other Ptolemy

Introduction

The main purpose of the preceding chapters was to present modern students with an outline of the Almagest in such a way that its inner structure would become clear. A secondary purpose was to express the fundamental concepts and relations of Ptolemaic astronomy in a way profitable to the study of Mediaeval astronomy. However, from the latter point of view it will be convenient to conclude with a few words on some other Ptolemaic works of particular importance to the Middle Ages.

It is true that in the history of astronomy Ptolemy was mainly the Ptolemy of the Almagest, and that for very obvious reasons. It was by far his largest astronomical treatise, in which all the planetary theories were constructed by means of geometrical models with parameters derived from actual observations. Moreover, the Almagest was composed with a disciplined rigour setting a very high standard for serious astronomical literature. Nobody can escape a feeling of profound respect and admiration towards a work of such classical beauty and strength. It is no wonder that the Almagest was able to mark the development of astronomy for centuries. In fact, Ptolemy's main work must be considered the ultimate source of practically all astronomy in the Western world until it was finally superseded through the efforts of Kepler and Newton.

Nevertheless, the astronomy of the Middle Ages contains several important lines of development starting from other works by Ptolemy. They were all written after the Almagest had been completed, presumably in order to answer some of the questions on which the earlier work remained silent. In this final chapter we shall give a brief summary of these works by 'the other Ptolemy', pointing out how some of them came to influence the course of astronomy to an extent which in some respects surpassed that of the Almagest itself.

The Planetary Hypotheses

First we meet the Ptolemy of the *Hypotheses*. This work has been rather badly handled by tradition. Only a part of Book I has been preserved in the original Greek, while Book II is known only from an Arabic translation.

The Hypotheses are dedicated to the same Syrus as the Almagest [I, 1; Hei 70]. Several references to the latter work indicate the order in which they were written so it is plausible that the Hypotheses are a kind of supplement to the Almagest. This is

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6_13,

corroborated by the preface, where Ptolemy says that while the Great Syntaxis gave a mathematical exposition of the celestial motions, he now wants to deal with them in a more general way, appealing more to the imagination. There are even some useful hints for instrument-makers who want to represent the whole universe by means of mechanical models: *As for the position and arrangements of the spheres* [...] *we shall use the simplest method, so that it will be easy to construct instruments – even if we depart a little from the correct proportions* [I, 2; Hei 73].

Among the simplifications touched upon here are a few cases of rounding off parameters to more convenient values; thus the eccentricity of the Moon is changed from $12^p;28$ to $12^p;30$. Other changes are without doubt due to new observations, e.g. the epicycle radius of Mercury is corrected from $r = 22^p;30$ to $22^p;15$ and the radius of the small circle of the Mercury theory from $3^p;0$ to $2^p;30$. In the apogees there are slight changes of 0°;17 for the Moon and 0°;1 for Jupiter. The only large change is that the top of the deferent of Saturn is placed 10° more to the East than in the Almagest.

More important from a theoretical point of view is the fact that some of the geometrical models are changed in structure, mainly in the theory of latitudes, which is greatly simplified, and, in fact, considerably improved.

Thus there is no oscillation of the deferent planes of Venus and Mercury (see p. 369). The two planes are given a constant inclination of 0°;10. The epicycle of Mercury is given a constant deviation of 6°;30 (not 6°;15) from the plane of the deferent, and that of Venus also a constant deviation of 3°;30 (not 2°;30). In the cases of the superior planets the maximum deviation is no longer different from the inclination. The values of each of these angles are 2°;30, 1°;30, and 1°;50 in the cases of Saturn, Jupiter, and Mars respectively. A most important consequence is that the planes of these epicycles are now parallel to the plane of the ecliptic. This was not the case in the Almagest (see page 365). From a Copernican point of view this is the only acceptable arrangement, the epicycle being for each model the circle reflecting the motion of the Earth around the Sun.

Among the simplifications introduced by Ptolemy in the Hypotheses is also his representation of the heavenly motions *by circles only, as if they were separated from their surrounding spheres* [I, 2; 73]. This is reflected in the terminology, where a circle is often denoted as a 'sphere'. As an example we quote: *Let us imagine a large, fixed circle through the centre of the world, and let it be called the sphere of the equator* [I, 3; 75], or the following: *For the sphere of the Sun we imagine a circle lying eccentrically* in the plane of the zodiac [I, 8; 81]. Such spheres were mentioned occasionally in the Almagest (see p. 167), but without details, and without playing any role in the development of the planetary theories. If Ptolemy had them in mind, he consciously abstracted from them, and it is understandable that he has often been described as a supporter of a purely mathematical astronomy without any real interest in the physical structure of the universe. It would seem, therefore, that the same attitude would prevail in the Hypotheses. But here one has to be careful. In fact Ptolemy says only that the motions of the heavens will be described without reference to spheres, without

implying that these could not be useful for other purposes. Thus one should not be deceived by the fact that in the Hypotheses all parameters are given in such units that the radii of all deferent circles are 60°. This was the case in the Almagest too, and is, of course, incompatible with a system of planetary spheres since it would make all the planetary models of roughly the same size.

In fact, when Ptolemy wants to treat of other things than the heavenly motions, the spheres are introduced in a very unambiguous way. This is clearly seen in the later part of the first Book of the Hypotheses, which is not extant in Greek and unknown to the Heiberg edition. Therefore one of the most important discoveries in the history of ancient astronomy in recent years was made when Hartner (1964) pointed to the probable existence of such a section of the work, whereafter Goldstein (1967) was able to identify it in an Arabic version which is now available in an English translation. It appears that in the Hypotheses Ptolemy not only wished to give a simplified exposition of the celestial motions, but that he also tried to describe the physical structure of the universe and determine the order and distance of all the planets, leading to the size of the world as a whole.

We remember that in the Almagest Ptolemy had found absolute distances of the Moon and the Sun only (see page 206 f.). The former was found from observations of lunar parallaxes, and the latter inferred by means of an argument utilizing observations of eclipses. But in the case of the planets such methods were impossible since no parallax could be measured. Therefore the distances of the five planets could not be found, and even their order could be determined only by various arguments of a more or less plausible character (see page 261). To these arguments the Hypotheses add one more, viz. that there are more different motions in the theories of the Moon and Mercury than in those of the other planets. This indicates that these two bodies are placed nearer than the other planets to the sphere of air, where there is plenty of motion. This is a curious argument, since Ptolemy obviously disregards the radical distinction between the heavens above and the elementary spheres below, which was a deeply rooted opinion in all Greek cosmology after Aristotle.

Now the size of the planetary system is computed in the Hypotheses from the assumptions, that 1) the order of the planets is the same as in the Almagest, and that 2) the model of each planet is a physical mechanism included in a spherical shell concentric with the Earth, and that finally 3) these shells or 'spheres' are fitted together in such a way that the outer surface of each one of them coincides with the inner surface of the next one without any intermediate space, either a void or a plenum: *This arrangement is most plausible, for it is not conceivable that there be in Nature a vacuum, or any meaningless and useless thing* [transl. Goldstein 1967, p. 8].

Guided by this principle Ptolemy now tries to solve the problem by the empirical data at his disposal. We know from the Almagest (see page 215) that the greatest and least distances of the Moon from the centre of the world are 64^r and 33^r respectively, where 1^r is the radius of the Earth. For the Sun, the Almagest gives a distance of 1210^r, without stating whether this is a maximum, mean, or minimum distance. In fact it must be understood as a mean distance, since the Hypotheses give the maximum and

minimum values as 1260^r and 1160^r respectively. This is consistent with the value of the eccentricity $e/R = 1/24$ found in the Almagest (see page 212 f.).

These numbers can be interpreted in the following way: When we move outwards from the centre of the universe the elementary spheres of Earth, Water, Air and Fire are all included in a sphere of radius 33^r. The mechanism of the lunar theory is comprised inside a sphere reaching from 33^r to 64^r, and that of the Sun inside a sphere from 1160^r to 1260^r. The problem is to fit the sphere of the remaining five planets into this scheme, using for each planet the ratio of its maximum and minimum distances as inferred from the geometrical models. In the Hypotheses these ratios are given as follows (cf. Hartner 1964, pp. 258 ff. and 266 ff.):

Mercury	88 : 34
Venus	104 : 16
Mars	7 : 1
Jupiter	37 : 23
Saturn	7 : 5

Ptolemy first fits the sphere of Mercury tightly around the sphere of the Moon. Thus the inner surface of the former must have a radius of 64^r, and the outer a radius of

$$64^r \cdot \tfrac{88}{34} = 166^r$$

In the same way we fit the sphere of Venus tightly around that of Mercury. This gives an outer radius of

$$166^r \cdot \tfrac{104}{16} = 1079^r$$

Already at this point a difficulty appears since the inner radius of the sphere of the Sun was found to be 1160^r. Accordingly there is a slip between the sphere of Venus and that of the Sun, contrary to what was supposed before. In Ptolemy's own words: *There is a discrepancy between the two distances which we cannot account for*; *but we were led inescapably to the distances which we set down* [transl. Goldstein, 1967, p. 7]. Now Ptolemy does not want to introduce a 'useless' sphere with a thickness of 81^r, whether it be a vacuum or whether filled with some celestial substance or aether. Instead he points out that the difficulty can be avoided if we diminish the distance of the Sun, or increase that of the Moon. This amounts to the same thing because of the way in which the distance of the Sun was inferred from that of the Moon (page 209). But this escape is only hinted at, no computation of the changed values being carried out in the Hypotheses. One can imagine that Ptolemy, after all, shrank from discarding parameters found from observations in order to save a principle which was as arbitrary to astronomy as it was agreeable to philosophy. He therefore proceeds with calculating the spheres of the superior planets, finding the exterior radius for the sphere of

$$\begin{aligned}\text{Mars} &= 1260^r \cdot \tfrac{6}{1} = 8820^r\\ \text{Jupiter} &= 8820^r \cdot \tfrac{37}{23} = 14187^r\\ \text{Saturn} &= 14187^r \cdot \tfrac{7}{5} = 19865^r\end{aligned}$$

The latter value is also the inner radius of the 8th sphere containing the fixed stars.

Having found the sizes of the spheres with one radius of the Earth as the unit of length, Ptolemy then calculates them in *stadii*, one *stadium* being a unit of length which is not defined. But the circumference of the Earth is given as 18 myriad *stadii*, or 180000 *stadii*, in agreement with the value given in Ptolemy's *Geography*. If we assume that Ptolemy knew the correct size of the Earth, we have accordingly

$$180000 \textit{ stadii} = 40000 \text{ km}$$

or

$$1 \textit{ stadium} = \text{ca. } 222 \text{ m}$$

We shall not go into these calculations, nor into Ptolemy's computation of the diameters of the actual planets from the value of the diameter of the Sun, found in the Almagest (see page 213) to be 11^r, combined with the more arbitrary assumptions that the planetary diameters are the following fractions of the diameter of the Sun

Venus	1/10
Jupiter	1/12
Mercury	1/15
Saturn	1/18
Mars	1/20

Thus the new-found section of the Hypotheses has revealed the very remarkable fact that it is in fact Ptolemy himself who is responsible for what history has called the Ptolemaic system of the world. He was not a purely mathematical astronomer interested only in the description of heavenly motions. He also felt it his duty as an astronomer to give an account of the physical structure of the universe, combining the traditional conception of a universe as composed of closely fitting spheres with his own theory of planetary motions. This gives him a new place in the history of astronomy and points to the conclusion that the often mentioned difference between a mathematical and a physical school of astronomers is smaller than we have been used to think[1]).

What strikes one most in this Ptolemaic universe is its small size compared with what we know to-day. In fact, Saturn has a maximum distance of about 20000 radii of the Earth, which is also the distance of the sphere of the fixed stars. This is only about 125 million kms, or less than the actual distance of the Earth from the Sun. Thus the Ptolemaic universe was small, regarded with modern eyes.

Ptolemy seems to have been aware that computations of this kind were, after all, less creditable than the numerical values derived from observations in the Almagest. Many of the conclusions in the Hypotheses are stated very carefully as hypothetical statements, like the following (relating to a computation of the volumes of the planets):

1) Duhem made much of these two ancient 'schools' in his *Système du Monde*, vol. 2, Chapters 10–11. In the Middle Ages the discussion was renewed and mathematical astronomy attacked by Aristotelian philosophers advocating a 'physical' theory of the universe; see Duhem, vol. 4, Chapters 8–9.

If all the distances have been given correctly then the volumes ⟨computed here⟩ *are also in agreement with what we have said* [transl. Goldstein, 1967, p. 9]. Logically much of the contents of the Hypotheses belong to another class than those of the Almagest. Perhaps this is one of the reasons why Ptolemy did not include them in his main astronomical work.

Having in the first Book of the Hypotheses described the general arrangement of the spheres, Ptolemy in the second Book turns to a detailed exposition of the mechanism contained inside each sphere *in agreement with what is proper to the nature of the bodies of the spheres and necessary for the principles proper to eternally unchanging beings* [II, 1; Hei 111].

This implies, first, a number of physical considerations. The spheres are supposed to be material, consisting of the celestial aether and completely penetrable to the light and influence of the various stars. They are perfectly round and smooth and revolve around their several poles without any resistance or constraint. They are not moved by any external forces, but by the particular star located inside each sphere. Each of the planets is supposed to possess a vital force imparting the correct motion to each of the moving parts of the mechanism. As in Aristotle, the origin of this conception is animistic: the spheres are moved and animated by living forces intimately connected with the spheres themselves, and not separated from them as in the cosmology of the later Middle Ages.

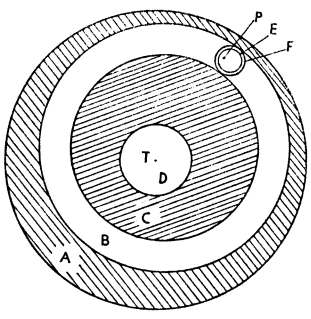

Fig. 13.1

The individual mechanisms of the various planets are described very carefully and illustrated by a number of figures, among which we select that showing the Saturn model as a typical example. The preceding Figure 13.1 shows a somewhat simplified version in which only the main features are included. Here A is an aethereal body, the outer surface of which is concentric with the Earth T while the inner surface is an eccentric sphere.

Inside A is located a spherical shell B both surfaces of which are concentric with another centre than T. Inside B is a third eccentric-concentric body C in the concentric hollow D of which the mechanism of Jupiter, and of all the rest of the planets, is placed. Saturn itself is found at P inside the solid sphere E revolving in a spherical shell F placed in a hollow in B.

The kinematics of this machine corresponds to that of the purely geometric Saturn model in the Almagest described in Chapter 9 above. The first 'sphere' A is responsible for the diurnal revolution of the whole system, and for the slow precessional motion relative to the fixed stars. The revolution of B is equivalent to that of the deferent, and E and F reproduce the motions of the epicycle.

We notice that there is no sphere corresponding to the equant circle, and therefore no body revolving with uniform mean angular velocity. Thus the non-uniform motion of the deferent sphere B is not produced by any kinematical device, but is caused directly by the vital force of the whole system. No wonder that this force has to be conceived as an 'intelligence'.

In the last parts of Book II Ptolemy calculates the number of spheres necessary to represent the whole system, including the sphere of the fixed stars and a sphere outside the latter acting as the Prime Mover of the Universe. The result is that the whole system demands at least 34 spheres containing a total of 22 aethereal bodies. This justifies Ptolemy's claim that he has succeeded in constructing a simpler system than any of his predecessors [II, 18; Hei 143].

The Hypotheses end with a section revealing how much the practical side of astronomy occupied Ptolemy's mind. First there is a reference to instruments by means of which the motions of the heavens can be simulated or illustrated, followed by a short description of a particular set of tables constructed by Ptolemy and appended to the Hypotheses as a means of determining planetary positions etc. by means of simple calculations. The description is sufficiently detailed to show that Ptolemy refers to what has become known as his *Handy Tables* (*Tabulae manuales*), to which we must now turn our attention.

The Handy Tables

The *Handy Tables* were dedicated to the same Syrus as the Almagest and the Hypotheses. They were provided with a set of *Canones*, or instructions on how to use them, written by Ptolemy himself and still extant in the original version. The tables themselves do not exist any longer in the form given to them by Ptolemy, but in a more or less revised version due to the astronomer Theon of Alexandria, who flourished in the second half of the 4th century A.D., the same who has left a commentary to the Almagest[2]).

Let us first consider the *Canones* to the tables. Ptolemy begins by stating that the tables are computed for the longitude of Alexandria and the era of Philippus (see page 127). Then follows a series of brief chapters showing how a number of astronomical

2) A critical edition of the *Handy Tables* is one of the principal *desiderata* of historians of Ancient and Mediaeval astronomy, Halma's edition and translation being unreliable in many details. For further information on these tables the reader is referred to the studies by Delambre (1817, II, p. 616 ff.), A. Rome (in his edition of Theon), B. L. van der Waerden (1953; 1957, col. 1823–27; 1958), W. D. Stahlmann (1960), and A. Aaboe (1960 a).

problems can be solved by means of the tables, only very simple arithmetical procedures being involved and no deep knowledge of planetary theory being necessary. But it is interesting that when Ptolemy treats the determination of planetary longitudes he supplements the arithmetical procedures by chapters describing a graphical solution based on the geometrical models known from the Almagest. Here a better knowledge of the theory is presupposed since the models are taken for granted and neither definitions nor numerical examples are given.

From the tables mentioned in the *Canones* one can get an impression of what Ptolemy's original collection of the Handy Tables must have contained. The following categories are mentioned.

1 Spherical-astronomical tables
2 Mean motion tables of the Sun, Moon, and planets
3 Tables of 'anomalies' (prosthaphairesis-angles)
4 Table of the Sun's declination
5 Tables of latitudes
6 Tables of stationary points
7 Tables of heliacal risings and settings
8 Parallax of the Moon
9 Tables of New and Full Moons
10 Eclipse tables for the Sun and Moon
11 A chronological table of the reigns of Kings

As for the tables themselves, we only possess Theon's versions, as already mentioned; but his revisions seem to have been slight and not too significant. So much can be said, that most of the tables are derived from those found in the Almagest although with some interesting changes. Thus the Almagest contained a set of mean motion tables with 18-year intervals. These are here replaced by tables with 25-year intervals which are more convenient when periods longer than a century are involved. Another innovation makes the tables still easier to use, viz. that in all tables where the independent variable is an angle the interval is reduced to 1° instead of the 3° or 6° usually found in the Almagest. Finally the exceedingly high, although illusory exactness of the Almagest-tables with their many sexagesimal places has been discarded and all values of dependent variables given in degrees and minutes only.

These are formal changes only, and in almost all cases the underlying theory is the same as in the Almagest. However, at one point the Handy Tables are different, and n fact more handy, viz. concerning the theory of latitude. Here the very complicated theory outlined above (see Chapter 12) has been simplified by the assumption that the deferent has a fixed position relative to the plane of the ecliptic with which it forms the constant angle i. If the planet P were found in the plane of the deferent at an angular distance (see Figure 13.2)

$$u = \lambda - \lambda_n - 90°$$

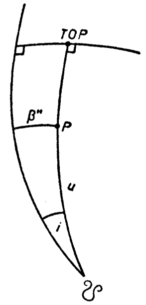

Fig. 13.2

from the top of the deferent, then it would have a latitude determined by

$$\sin \beta'' = \sin i \cdot \sin u$$

or, approximately

$$\beta'' = i \cdot \sin u$$

since the angle i is small. Actually the values of i are the following:

$$\left.\begin{array}{l} 2^\circ;30 \text{ for Saturn} \\ 1^\circ;30 \text{ for Jupiter} \\ 1^\circ;0 \text{ for Mars} \end{array}\right\} \text{as in the Almagest}$$

0°;10 for Venus
0°;10 for Mercury

Usually the planet P will be found somewhere on the epicycle outside the deferent plane. In the Almagest the epicycle performed a rather intricate tilting motion with a variable deviation j from the plane of the deferent. This complication has been done away with here and the epicycle is supposed to be parallel to a fixed direction in space (although not to the ecliptic, as in the Hypotheses, cf. page 392). Suppose that the epicycle radius CP forms the angle w with a line through C having a fixed direction perpendicular to the nodal line. In the notation of the following Figure 13.3 we then have

$$\sin \beta' = \frac{r \sin j \cdot \cos w}{\Delta}$$

where Δ is the distance between the Earth and the planet.

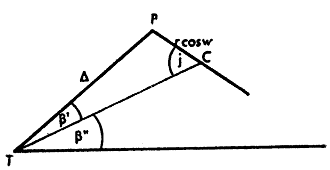

Fig. 13.3

The latitude of the planet is then expressed as a sum of two components

$$\beta = \beta' + \beta''$$

similar to the sum (12.40) of the three components by which the latitudes of the inferior planets were expressed in the Almagest.

The deviations j used in the Handy Tables are for

$$\left.\begin{array}{ll}\text{Saturn} & 4°;30 \\ \text{Jupiter} & 2°;30 \\ \text{Mars} & 2°;15\end{array}\right\} \text{ as in the Almagest}$$

$$\left.\begin{array}{ll}\text{Venus} & 3°;30 \\ \text{Mercury} & 6°;30\end{array}\right\} \text{ as in the Hypotheses}$$

The Tetrabiblos

With his Handy Tables Ptolemy simplified the calculation of planetary positions, eclipses, etc. to the utmost possible extent. There can be no doubt that the tables were composed for the benefit of people who had to perform astronomical calculations in great numbers, but without any high accuracy. Only one class of scientists fulfils these conditions, namely the astrologers. That Ptolemy was aware of the needs of the latter and was, in fact, one of their number himself, is seen from the work he called *Apotelesmatika*, but which has come down to us with the title *Tetrabiblos*, or *Liber quadripartitus*. This was in the Middle Ages the best known among Ptolemy's writings, and is still conserved in a great number of Greek MSS and numerous Latin versions. It has been printed several times and was the first work by Ptolemy of which an English translation was made.

The Tetrabiblos is also dedicated to Syrus. From this follows perhaps that the Almagest, the Hypotheses, the Handy Tables, and the Tetrabiblos were planned as a collection comprising all parts of astronomy. One might wonder why Ptolemy included a purely astrological work in it, and some authors have indeed ascribed the Tetrabiblos to another author, thinking that the Ptolemy who wrote the Almagest could not be responsible for a work of superstition. But there is no doubt that the book is genuine, and that Ptolemy did not regard its subject matter as superstitious. On the other hand he is no naïve supporter of all kinds of astrology. In fact, in many ways the first three chapters of the Tetrabiblos reflect the long debate on the scientific status of astrology going on among philosophers in Antiquity. Ptolemy begins by pointing out that there are two different kinds of prediction which the astronomer is able to make. First, he can predict the motions of the Sun, Moon, and stars relative to each other and to the Earth. How this is done is shown mathematically in the Almagest. But second, the astronomer can also use the various aspects of the planets for predicting their influence on other bodies, and mainly upon the Earth. How such predictions are carried out is shown in the Tetrabiblos, which thus appears as an

astrological complement to the purely astronomical Almagest. In the following we shall denote the first kind of predictions as astronomical, and the second as astrological[3]), although Ptolemy does not distinguish them by different words.

In a way, both astronomy and astrology are parts of the same art of making predictions of, and from, the stars, but that does not imply that they have the same status and attain the same certainty. The Almagest gave a mathematical theory of celestial motions which nobody can doubt 'unless he is blind'. This is in agreement with Ptolemy's general opinion on mathematics (page 29). Astrology is, on the other hand, a much more philosophical discipline. Moreover, it investigates individual phenomena happening to material bodies in the elementary world. Therefore it is tainted with the weakness and unpredictability inherent in such bodies. This explains why its conclusions have been doubted by some philosophers regarding them as incomprehensible and thus useless. Accordingly Ptolemy underlines that astronomy must come before astrology because the conclusions of the former are certain while those of the latter are only possible.

It is obvious that Ptolemy attaches a certain importance to the critics of astrology and feels the need of giving it some kind of rational foundations. This is done in Chapters 2 and 3, where he deals with the feasibility and usefulness of astrology. Here we must remember that originally astrology was foreign to Greek thought. It represented an Oriental influence combated by many different schools[4]). The sceptics of the new Academy headed by Carneades (ca. 214–129 B.C.) attacked its dogmatism and anthropomorphism. Physicists attacked it as a field for frauds and charlatans perverting true astronomy into superstition. Moralists denounced its fatalism, and theologians its dogmas. Nevertheless, astrology was able to permeate the Hellenistic world, with the Stoic philosophers as its chief advocates. The main figure was the Syrian Posidonius of Rhodes (ca. 135–50 B.C.), Cicero's teacher and a prolific author, who in the Hellenistic period had almost the same status as Aristotle in classical Athens. Of his works only fragments remain, but they were the direct inspiration of the great didactic poem by Manilius which transmitted astrological ideas throughout the Latin world and on to the Middle Ages. As Boll (1894, p. 131 ff.) has shown, Ptolemy's defense of astrology is derived directly from Posidonius, which proves once more how Ptolemy as a thinker was impressed by Stoic philosophy (see page 31). Actually Boll's argument rests upon the fact that many of Ptolemy's reasons are found in parallel passages in Cicero, Philo, Cleomedes, and Manilius. Therefore they must be derived from the one source these authors had in common, namely Posidonius.

Ptolemy first refers to the obvious fact that the Sun regulates the seasons, thus influencing all forms of life on the Earth, and that the Moon governs the tides. This is

3) This distinction is still preserved in early Mediaeval works, for instance in Isidore of Seville's *Libri Etymologiarum* (III, 27). Many later authors caused much confusion by interchanging the meaning of the two terms.

4) There is no comprehensive study of ancient astrology taking the results of modern research into account. Two older works are still useful, viz. A. Bouché-Leclerq, *L'astrologie grecque*, Paris 1899, and F. Cumont, *Astrology and Religion among the Greeks and Romans*, London, 1912 (Reprinted New York, 1960).

obviously true; but now the principle of the uniformity of nature leads Ptolemy astray. He maintains that also the water in rivers increases and diminishes with the light of the Moon, and that plants and animals are sympathetic to it because of their moist nature. Now the Sun and Moon are only two representatives of the heavenly bodies, and all the rest must be supposed to cause similar effects. Even the fixed stars are assumed to have a direct influence on meteorological conditions. These many influences can mutually interfere, so that their total effect on a particular phenomenon is determined by the positions of the heavenly bodies relative to each other, i.e. by their various aspects. The argument continues by stating that the influence of the Sun is felt even by animals, and that farmers and herdsmen are able to make predictions from the stars based upon repeated everyday observations. The same is the case with sailors. On the other hand, the more intricate predictions are possible only for people able to observe and describe the aspects of the heaven in greater detail, i.e. for professional astronomers.

Having thus established a positive basis for astrology, Ptolemy tries to refute some of the criticisms levelled against it. That charlatans and impostors have been unable to make true predictions proves nothing but that they are uninstructed and take too few of the relevant circumstances into account. No science is discredited because ignoramuses misuse it. Admittedly astrology is a difficult science, but if the astrologer does not promise too much, and if he has a solid astronomical background, one should not deny the possibility of making plausible predictions, just as we do not dismiss the whole science of navigation because pilots sometimes err.

Besides being possible, astrology is also beneficial. Again Ptolemy begins with a number of positive reasons. First he points to the general pleasure and satisfaction connected with any true insight into things, both human and divine. (The inference is that, everything considered, Ptolemy attaches more importance to delight in knowledge as such than to its practical utility.) But astrology is particularly useful as a means of knowing what is harmful or good for both body and soul, enabling us to predict not only occasional diseases or the length of life, but also external circumstances which have a direct and natural connection with the original gifts of nature, such as property and marriage in the case of the body, and honour and dignity in the case of the soul.

Among the objections to the usefulness of astrology is that it does not help people to become rich or famous, but only reveals their unavoidable fate, which were better if it remained hidden. To the first part of this rather materialistic objection Ptolemy remarks that the same is true of all philosophy, since knowledge as such does not lead to material gains. To the latter part he replies as a true Stoic philosopher that the foreknowledge of the unavoidable is useful to the soul, which can thus rejoice in future pleasures or compose itself to meet future pains with calm and steadiness.

This defense of astrology is an intelligent piece of reasoning which shows that Ptolemy was no uncritical adherent of a doctrine which to later times appears as mere superstition. On the one hand he tried to find a rational basis making the possibility

of predictions acceptable. On the other hand he was completely aware that the empirical foundation of astrology was much weaker than that of astronomy, resting as it did upon everyday experiences in contrast to the precise observations of the Almagest, and upon correlations of a not too satisfactory character. He thus had to place astrology at a lower scientific level than astronomy, and it is worth noticing that astrology found no place whatever in the Almagest (cf. Kattsoff, 1947, p. 18). To Ptolemy the precise mathematical theories of the Almagest not only represent the highest level astronomy was able to attain, but also the very summit of human knowledge. Not all his followers retained this order; many Arab and Mediaeval astronomers considered planetary theory as a simple introduction to the art of prediction, thus separating Ptolemy's results from the spirit in which they had been achieved.

As it appears from its usual title, the *Tetrabiblos* contains four books, each dealing with a special part of astrology. Book I is about the power of the planets, the fixed stars, and the signs of the zodiac, and defines the various astrological concepts and terms. Book II is mostly about the public influence of stars and eclipses on various countries and their inhabitants. Book III is about nativities, i.e. predictions of individual fortunes from the position of the stars at the moment of birth. Finally, Book IV is about the external circumstances in life, like fortune marriage, children, friends, travels, ending with the chapter on the quality of death. We shall not here go into the methods by which such things were predicted, but only make the concluding remark that Ptolemy's Tetrabiblos, with its clear and precise structure, not only rehabilitated astrology in Hellenistic times, but also became the ultimate source of almost all later European astrology.

The Analemma and the Planisphaerium

The last two works which we are going to consider here are the *Analemma* and the *Planisphaerium*. The first of these is preserved only in a Latin translation made directly from the Greek by William of Moerbecke. Also the Greek text of the second treatise is lost, but an Arabic version by Maslama al-Majrīṭī was translated into Latin by Herman the Dalmatian in 1143. These versions are the basis of the modern editions.

The two small treatises deal with related topics, namely the solution of problems in spherical astronomy by means of geometrical constructions. Such methods were already applied long before Ptolemy's time and used for the construction of sundials. Thus an analemma for that purpose is described by Vitruvius (about 25 B.C.) and a similar one, but of a more general type, by Heron (about 62 A.D.), who used it for determining the distance between two localities with known longitudes and latitudes (see Drachmann 1950, and O. Neugebauer 1938–39).

Ptolemy's analemma was drawn upon a circular table on which the lines commonly used were painted, while others could be drawn in a thin coat of wax and easily

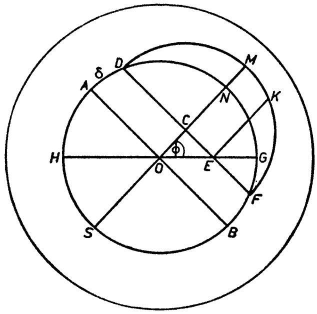

Fig. 13.4

erased again (Figure 13.4). A circle concentric with the table represents the meridian, and the diameter AOB the line of intersection between the plane of the meridian and that of the celestial equator. N and S are the poles, and GOH is the horizon. Then angle GON $= \varphi$ is the geographical latitude. In other words, the figure drawn on the board is a p a r a l l e l p r o j e c t i o n of some of the main circles on the heavenly sphere.

The use of this analemma for solving spherical-astronomical problems depends on the fact that a number of other circles on the heavenly sphere can be drawn on the board by means of parallel projection, or by folding them down upon it. As an example (already found in Vitruvius, *De Architechtura*, IX, 7) we can find the length of the day at a certain date on which the declination of the Sun is $\delta =$ AD. Then the line DCEF is the projection of the diurnal circle of the Sun which is folded down as DMF. A line through E perpendicular to DEF intersects this circle at K. Then DMK is half the diurnal arc of the Sun, and the length of the day in equinoctial hours is twice this arc divided by $\frac{1}{6}$ of the arc DM.

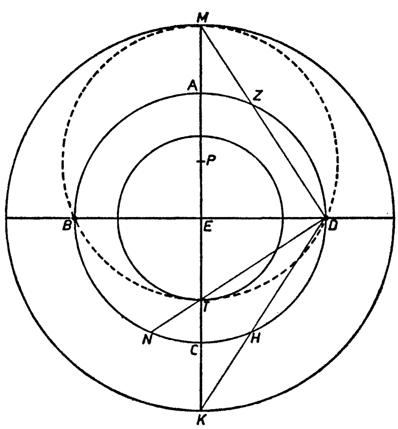

Fig. 13.5

In the *Planisphaerium* – which is dedicated to Syrus – Ptolemy describes how the celestial sphere can be projected in a different way, namely by stereographic (central) projection from the southern pole of the heavens upon the plane of the equator (Figure 13.5). A circle ABCD represents the celestial equator, and its centre E the North pole. At both sides of C the points N and H are marked so that CN = CH = ε where ε is the obliquity of the ecliptic. The lines DN and DH intersect EC at T and K, and circles are drawn with centre E and radii ET and EK respectively.

Ptolemy now maintains that the circle through T is the projection of the tropic of Cancer, while that through K represents the tropic of Capricorn. This is easily seen if we regard the circle ABCD as the meridian, AC as the equator, and D as the southern pole. Then it is obvious that all circles on the sphere parallel to the equator will be projected into circles in the plane concentric with the equator.

The next problem is how the ecliptic is projected upon the plane of the equator. Ptolemy states that the ecliptic is represented by the dotted circle drawn with MT as diameter around the centre P. This is proved by the following argument: The dotted circle passes through T and M, and corresponds accordingly to a circle through T on the tropic of Cancer and M on the tropic of Capricorn, while the rest of it lies between the two tropics. Furthermore, the line DM intersects the equator at Z so that AZ = CH = ε. It follows that ZDHN is a semicircle. Therefore angle ZDN = 90° = angle MDT. But MT is a diameter of the dotted circle, so that D is a point of its circumference. That is the case also of B. In other words, the dotted circle passes through opposite points of the equator. Consequently it represents a great circle having precisely one point in common with each of the tropics, i.e. the ecliptic.

This proof holds true under the assumption that the ecliptic is projected upon the plane of the equator as a circle. This Ptolemy does not prove, presumably because it was well known in his time. In fact stereographic projection has the property that any circle on the sphere is projected as a circle or a straight line. We do not know for certain who first realized or proved this theorem, but it seems to have been known already to Hipparchus.

Ptolemy continues by proving that any straight line through E cuts any circle through B and D in two points representing diametrically opposed points on the heavenly sphere. This leads immediately to the construction of a set of circles each of which bisects both the equator and the ecliptic; therefore the set represents all possible horizons. It is proved that they all pass through A and C.

The importance of the Planisphaerium is apparent from a remark in Cap. 14, where Ptolemy mentions the Horoscopic Instrument provided with an *aranea* (i.e. spider). This indicates that he knew an instrument based on a stereographic projection of the heavenly sphere. This could be either the anaphoric clock described by Vitruvius, or the plane astrolabe, which is certainly older than the earliest known description by Johannes Philoponos (6th century A.D.). This seems to indicate that the Planisphaerium was Ptolemy's own contribution to the theory of such instruments (cf. Drachmann 1954, and O. Neugebauer 1949).

Other Works

A few other works by Ptolemy are of importance to the history of astronomy. We shall mention them only briefly, considering that they were unknown to the Latin Middle Ages and without any influence on the astronomical tradition in Europe.

One such work is the *Phaseis*, or *Parapegma* (= weather calendar). It deals with the phases, or visibility periods of the fixed stars (cf. Almagest VIII, 4–6) and how they can be used for astrological weather predictions. The treatise has been analysed by Vogt (1920). It was not translated into Latin until 1592 when Federigo Bonaventura published it at Urbino. On the other hand, a work known as *Karpos* in Greek and *Centiloquium* in Latin was extremely popular in the Middle Ages and ascribed to Ptolemy although it is now known to be spurious. It is a disorderly compilation of one hundred astrological aphorisms.

Of much greater importance was Ptolemy's great *Geography*, a description of the whole world as it was known, or supposed to be known, by geographers of Hellenistic times. It contains geographical coordinates of about 8000 localities and is of particular interest to mathematics because of the cartographical principles developed by Ptolemy as a foundation for the many maps with which the work was provided (see Neugebauer 1959 b; cf. Fischer 1932). The *Geography* was translated into Latin at the beginning of the 15th century. The first printed version (Vicenza, 1475) was followed by numerous other editions. The work provoked an enormous outburst of cartographical interest of immediate consequence to the Age of Discovery.

One of Ptolemy's minor works was of a very particular nature, viz. the stele he erected in the 10th year of Antoninus in the town of Canopus, East of Alexandria (where he may well have lived, see page 12). It was dedicated to the God Soter (the Saviour) and was provided with an inscription listing numerical values of all the important parameters of Ptolemaic astronomy. Most of them agree with the values quoted in the Almagest, and the few changes are less drastic than in the *Handy Tables* or the *Planetary Hypotheses* (see v. d. Waerden 1957, col. 1818–23). Like these works. the inscription is a testimony to Ptolemy's persistent attempts to improve upon his own results, long after the *Almagest* was finished. The monument itself perished long ago.

Envoi

The works of Ptolemy mentioned in this Epilogue had an enormous influence on the astronomy of later times, although often along a very indirect and tortuous road.

The Hypotheses gave the Ptolemaic picture of the universe of spheres. Long forgotten, it turned up in the Middle Ages through the astronomical writings of Ibn al-Haitham (Alhazen), giving rise to a new battle between 'mathematical' and 'physical' astronomers, and retaining its authority until the spheres at long last vanished into thin air under the attack of Tycho Brahe.

In the same way the Handy Tables became the paradigm for scores of similar collections made by Arabic astronomers together with not a few Latin ones. Practically all Mediaeval tables go back to them, if not with respect to fundamental parameters, then at least in their general contents and lay-out.

In a way, the Tetrabiblos was the most influential of all Ptolemy's works, if we are to judge by the number of astrological writings stemming from it, first among the Arabs, and later among the Latins towards the end of the Middle Ages and in Renaissance times. While the Almagest remained forever a work for the select few readers able to cope with its mathematics, the doctrine of the Tetrabiblos and its descendants impressed itself upon the minds of everyman, as another proof that popularity and intrinsic value are often inversely proportional quantities.

Finally, the Analemma and the Planisphaerium fostered still another branch of Mediaeval astronomy – the science of instruments. From the Analemma stems the theory of sundials, which, as the science of gnonomics, gave rise to more literature before, and particularly after the invention of printing than most other astronomical subjects (ordinary calendars excepted). And from the Planisphaerium came both the astrolabe and the art of making stellar maps. The former is of extreme importance as the first precision instrument in science, forcing instrument makers to display their skill on more and more sophisticated types until they were well equipped to provide a Tycho Brahe or a William of Hessen with the instruments they needed. The importance of the latter for the general evolution of cartography and navigation need not be stressed here.

Thus the other Ptolemy proves just as influential as the author of the Almagest. But notwithstanding the passing fame of the Tetrabiblos, or the charming elegance of the minor works, the Great Syntaxis will remain the best testimony to a scientific achievement not often met with in the history of mankind.

APPENDIX A

Dated Observations in the Almagest

The analysis and criticism of ancient observations lie outside the scope of this book which is concerned more with the theoretical structure of Ptolemaic astronomy than with the reliability of its empirical basis. Therefore the following list has been appended only for purposes of reference. It contains all the observations mentioned in the Almagest to which a precise date can be ascribed, excluding, for instance, a number of Hipparchian observations to which the references are vague [see III, 1; Hei 1, 196]. The 94 entries are numbered and arranged in chronological order according to the Julian dates calculated by Manitius (line 1). These dates may in some cases be one day wrong. The second line quotes the date of the observation as given in the Almagest with a reference to the place where it is first mentioned. Line 3 gives the time of the day as indicated by Ptolemy. Then follows the nature and the immediate result of the observation (line 4), and in line 5 it is indicated how it was used by Ptolemy in the general theoretical framework of his book. In some cases the final line 6 gives references to authors who have discussed the observation in question, or corrected the date and time given by Manitius.

1 **721 B.C. March 19** [Man 1, 219].
Mardokempad 1 Thoth 29/30 [IV, 6; Hei 1, 302].
A little more than one hour after sunrise.
Babylon.
Beginning of total *lunar eclipse* (E_1 page 170).
Used with **2** and **3** for determining the epicycle radius of the Moon (page 171 ff.).
Boll 1909, col. 2353 / Ginzel 1889, page 232 / Newcomb, page 35 / Zech 1851, page 13.

2 **720 B.C. March 8** [Man 1, 220].
Mardokempad 2 Thoth 18/19 [IV, 6; Hei 1, 303].
Midnight.
Babylon.
Maximum of partial *lunar eclipse*, 3 digits southern (E_2 page 170).
Used with **1** and **3**; also with **69** for correcting ω_t of the Moon (page 180); and with **6** for calculating the radices of the lunar theory (page 182).
Boll 1909, col. 2353 / Ginzel 1899, page 233 / Newcomb, page 36 / Zech 1851, page 13.

3 **720 B.C. Sept 1** [Man 1, 220].
Mardokempad 2 Phamenoth 15/16 [IV, 6; Hei 1, 303].
After moonrise.
Babylon.

O. Pedersen, *A Survey of the Almagest*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-0-387-84826-6,

Beginning of partial (more than 6 digits northern) *lunar eclipse* (E_3 page 170).
Used with **1** and **2**.
Boll 1909, col. 2353 / Ginzel 1899, page 233 / Newcomb, page 36 / Zech 1851, page 13.

4 **621 B.C. April 22** [Man 1, 306].
Nabopolassar 5 (Nabonassar 127) Athyr 27/28 [V, 14; Hei 1, 418].
Towards the end of the 11th night hour.
Babylon.
Beginning of partial (3 digits southern) *lunar eclipse.*
Used with **5** for finding the minimum apparent diameter of the Moon (page 208).
Boll 1909, col. 2353 / Ginzel 1899, page 233 / Newcomb, page 36 / Zech 1851, page 13.

5 **523 B.C. July 16** [Man 1, 308].
Kambyses 7 (Nabonassar 225) Phamenoth 17/18 [V, 14; Hei 1, 419].
One hour before midnight.
Babylon.
Maximum of partial (6 digits northern) *lunar eclipse.*
Used with **4** (page 208).
Known from Babylonian records, see Kugler 1907, pp. 61–75.
Boll 1909, col. 2354 / Ginzel 1899, page 233 / Newcomb, page 37 / Newton 1970, page 136 and 140 f. / van der Waerden 1958 b / Zech 1851, page 13 and 27–30.

6 **502 B.C. Nov 19** [Man 1, 241].
Darius 20 Epiphi 28/29 [IV, 9; Hei 1, 132].
When $6\frac{1}{3}$ equinoctial hours had passed after nightfall.
Babylon?
Maximum of partial (3 digits southern) *lunar eclipse* (E_4 page 170).
Used with **2** for finding the radix of the lunar motion in latitude.
Boll 1909, col. 2354 / Ginzel 1899, page 233 / Newcomb, page 37 / Newton 1970, page 140 f. / van der Waerden 1958 b / Zech 1851, page 13.

7 **491 B.C. April 25** [Man 1, 239].
Darius 31 (Nabonassar 256) Tybi 3/4 [IV, 9; Hei 1, 329].
In the middle of the 6th night hour.
Babylon?
Maximum of partial (2 digits southern) *lunar eclipse* (E_5 page 170).
Used with **54** for correcting ω_a of the Moon (page 181).
Boll 1909, col. 2354 / Ginzel 1899, page 233 / Newcomb, page 37 / Zech 1851, page 14.

8 **432 B.C. June 27** [Man 1, 143].
Apseudes (Archont of Athens) Phamenoth 21 [III, 1; Hei 1, 205].
Morning.
Athens – Meton and Euctemon, or their pupils.
Time of *summer solstice* (S_1, page 130).
Used with **16** and **92** for determining the tropical year (page 132).
Rome 1937, page 216.

9 **383 B.C. Dec 23** [Man 1, 247].
Phanostratos (Archont of Athens) Poseidon s.d. (Nabonassar 366 Thoth 26/27) [IV, 11; Hei 1, 340].
One half hour before the night was over.
Babylon?
Beginning of small *lunar eclipse* (E_6 page 170).
Used with **10** and **11** by Hipparchus for determining the epicycle radius of the Moon on the eccentric hypothesis (page 177).
Boll 1909, col. 2356 / Ginzel 1899, page 233 / Newcomb, page 37 / Newton 1970, page 141 f. / van der Waerden 1958 b / Zech 1851, page 14.

10 **382 B.C. June 18** [Man 1, 248].
Phanostratos (Archont of Athens) Skirophorion s.d. (Nabonassar 366 Phamenoth 24/25) [IV, 11; Hei 1, 341].
In the advanced first hour of the night.
Babylon?
Beginning of partial *lunar eclipse* lasting three hours (E_7 page 170).
Used with **9** and **11**.
Boll 1909, col. 2356 / Ginzel 1899, page 233 / Newcomb, page 38 / Zech 1851, page 14.

11 **382 B.C. Dec 12** [Man 1, 249].
Euandros (Archont of Athens) Poseidon I s.d. (Nabonassar 367 Thoth 16/17) [IV, 11; Hei 1, 342].
In the advanced fourth hour of the night.
Babylon.
Beginning of total *lunar eclipse* (E_8 page 170).
Used with **9** and **10**.
Boll 1909, col. 2356 / Ginzel 1899, page 233 / Newcomb, page 38 / Zech 1851, page 14.

12 **295 B.C. Dec 21** [Man 2, 27].
I Calippus 36 Poseidon 25 (Nabonassar 454 Phaophi 16) [VII, 3; Hei 2, 32].
Beginning of the 10th hour of the night.
Alexandria – Timocharis.
Conjunction of Moon and β Scorpii.
Used in the theory of precession.
Fotheringham and Longbottom 1915, page 384 and 386.

13 **294 B.C. March 9** [Man 2, 24].
I Calippus 36 Elaphebolion 15 (Nabonassar 454 Tybi 5) [VII, 3; Hei 2, 28].
Beginning of the third hour of the night.
Alexandria – Timocharis.
Occultation of α Virginis by the Moon.
Used with **15** for determining the rate of precession (page 247).
Fotheringham and Longbottom 1915, page 384 and 386 / Knobel 1915.

14 **283 B.C. Jan 29** [Man 2, 22].
I Calippos 47 Anthesterion 8 (Nabonassar 465 Athyr 29/30) [VII, 3; Hei 2, 25].
End of the third hour of the night.
Alexandria – Timocharis.
Occultation of the Pleiades by the Moon.
Used with **51** for determining the rate of precession (page 247).
Fotheringham and Longbottom 1915, page 384 and 387.

15 **283 B.C. Nov 9** [Man 2, 25].
I Calippos 48 Pyanepsion 25 (Nabonassar 466 Thoth 7/8) [VII, 3; Hei 2, 29].
When $9\frac{1}{2}$ hour of the night had passed.
Alexandria – Timocharis.
Conjunction of α Virginis and the Moon.
Used with **13** (page 247).
Encke 1859 determined the date as Nov 8 and found the minimum distance to be 1′20″.
Fotheringham and Longbottom 1915, page 384 and 387.

16 **280 B.C.** [Man 1, 145].
I Calippus 50, end of the year [III, 1; Hei 1, 207].
No time of the day recorded.
Alexandria? – Aristarchus.
Time of *Summer solstice* (S_2 page 130).
Used with **8** and **92** (page 132) and by Hipparchus with **44**.

17 **272 B.C. Jan 18** [Man 2, 199].
Dionysius 13 Aegon 26 (Nabonassar 476 Athyr 20/21) [X, 9; Hei 2, 352].
Early Morning.
Alexandria?
Occultation of β Scorpii by Mars at a longitude of 212°;15.
Used for 'testing' the Mars theory.
Man 2, 406, Note 6.

18 **272 B.C. Oct 12** [Man 2, 167].
Philadelphus 13 (Nabonassar 476 Mesore 17/18) [X, 4; Hei 2, 310].
During the 12th hour of the night.
Alexandria – Timocharis.
Conjunction of η Virginis and Venus at a longitude of 154°;10 (V_1 page 299).
Used with **80** for determining ω_a of Venus (page 307); and for determining the radices of the Venus theory (page 308).

19 **272 B.C. Oct 16** [Man 2, 168].
Philadelphus 13 Mesore 21/22 [X, 4; Hei 2, 311].
Morning.
Alexandria – Timocharis.
Longitude of Venus = 158°;50.
Used to prove that **18** was not a maximum elongation.

20 **265 B.C. Nov 15** [Man 2, 150].
Dionysius 21 Scorpion 22 (Nabonassar 484 Thoth 18/19) [IX, 10; Hei 2, 288].
Morning.
Alexandria?
Longitude of Mercury = 213°;20 (M_1 page 310).
Used with **84** to correct ω_a of Mercury (page 324); also for determining the radices of the Mercury theory (page 324).

21 **265 B.C. Nov 19** [Man 2, 151].
Dionysius 21, Scorpion 26 (Nabonassar 484 Thoth 22/23) [IX, 10; Hei 2, 289].
Morning.
Alexandria?
Longitude of Mercury = 214°;05.
Used to prove that **20** was not a maximum elongation.

22 **262 B.C. Feb 12** [Man 2, 132].
Dionysius 23 Hydron 21 (Nabonassar 486 Choiac 17/18) [IX, 7; Hei 2, 264].
Morning.
Alexandria?
Position of Mercury relative to δ Capricorni giving a maximum western elongation of 25°;50 at a longitude of 292°;20 (M_2 page 310).
Used with **23** and **25** for determining the motion of the apsidal line of Mercury (page 312).
Czwalina 1959.

23 **262 B.C. April 25** [Man 2, 133].
Dionysius 23 Tauron 4 (Nabonassar 486 Mechir 1) [IX, 7; Hei 2, 265].
Evening.
Alexandria?
Maximum eastern *elongation* 24°;10 *of Mercury* at a longitude of 53°;40 (M_3 page 310).
Used with **22** and **25**.
Czwalina 1959.

24 **262 B.C. Aug 23** [Man 2, 134].
Dionysius 24 Leonton 28 (Nabonassar 486 Payni 30) [IX, 7; Hei 2, 267].
Evening.

Alexandria?
Position of Mercury relative to α Virginis giving a maximum eastern elongation of 21°;40 at a longitude of 169°;30 (M_4 page 310).
Used with **26** and **28** for determining the motion of the apsidal line of Mercury (page 312).
Czwalina 1959.

25 **257 B.C. May 28** [Man 2, 133].
Dionysius 28 Didymon 7 (Nabonassar 491 Pharmuti 5/6) [IX, 7; Hei 2, 265].
Evening.
Alexandria?
Position of Mercury relative to α and β Geminorum leading to a maximum eastern elongation of 26°;30 at a longitude of 89°;20 (M_5 page 310).
Used with **22** and **23** (page 312).
Czwalina 1959.

26 **245 B.C. Nov 19** [Man 2, 135].
Chaldaic era 67 Apellaios 5 (Nabonassar 504 Thoth 27/28) [IX, 7: Hei 2, 268].
Morning.
Babylon?
Position of Mercury relative to β Scorpii giving a maximum western elongation of 22°;30 at a longitude of 212°;20 (M_6 page 310).
Used with **24** and **28** (page 312).
Czwalina 1959.

27 **241 B.C. Sept 4** [Man 2, 223].
Dionysius 45 Parthenon 10 (Alexander 83 [= Nabonassar 507] Epiphi 17/18) [XI, 3; Hei 2, 386].
Dawn.
?
Occultation of δ Cancri by Jupiter at a longitude of 97°;33.
Used for 'testing' the Jupiter theory.

28 **237 B.C. Oct 30** [Man 2, 135].
Chaldaeic era 75 Dios 14 (Nabonassar 512 Thoth 9/10) [IX, 7; Hei 2, 267].
Early morning.
Babylon?
Position of Mercury giving a maximum western elongation of 21° at a longitude of 194°;10 (M_7 page 310).
Used with **24** and **26** (page 312).
Czwalina 1959.

29 **229 B.C. March 1** [Man 2, 247].
Chaldaeic era 82 Xanthikos 5 (Nabonassar 519 Tybi 14) [XI, 7; Hei 2, 419].
Evening.
Babylon?
Position of Saturn relative to γ Virginis at a longitude of 159°;30 (P_5 page 273).
Used with **74** for 'testing' the Saturn theory (page 289 f.).

30 **201 B.C. Sept 22** [Man 1, 251].
II Calippus 54 (Nabonassar 546) Mesore 16 [IV, 11; Hei 1, 344].
Middle of the third hour [of the night].
Alexandria.
End of a *lunar eclipse* lasting 3^h (page 170).
Used with **31** and **32** by Hipparchus for determining the epicycle radius of the Moon on the epicyclic hypothesis (page 177).
Boll 1909, col. 2358 / Ginzel 1899, page 233 / Newcomb, page 38 / Newton 1970, page 137 / Zech 1851, page 15.

31 **200 B.C. March 19** [Man 1, 252].
II Calippus 54 (Nabonassar 546) Mechir 9 [IV, 11; Hei 1, 345].
$5\frac{1}{3}$ civil hours after sunset.
Alexandria.
Beginning of a total *lunar eclipse* (E_{10} page 170).
Used with **30** and **32** (page 177).
Boll 1909, col. 2358 / Ginzel 1899, page 233 / Newcomb, page 39 / Zech 1851, page 15.
32 **200 B.C. Sept 12** [Man 1, 252].
II Calippus 55 (Nabonassar 547) Mesore 5 [IV, 11; Hei 1, 346].
$6\frac{2}{3}$ civil hours after sunset.
Alexandria.
Beginning of a total *lunar eclipse* (E_{11} page 170).
Used with **30** and **31** (page 177).
Boll 1909, col. 2358 / Ginzel 1899, page 233 / Newcomb, page 39 / Zech 1851, page 15.
33 **174 B.C. May 1** [Man 1, 350].
Philometor 7 (Nabonassar 574) Phamenoth 27/28 [VI, 5; Hei 1, 477].
From the beginning of the 8th to the end of the 11th hour [of the night].
Alexandria.
Duration of partial *lunar eclipse*, 7 digits northern.
Used with **42** for determining the maximum apparent diameter of the Moon (page 227).
Boll 1909, col. 2358 / Ginzel 1899, page 233 / Newcomb, page 39 / Zech 1851, page 15.
34 **162 B.C. Sept 27** [Man 1, 134].
III Calippus 17 Mesore 30 [III, 1; Hei 1, 195].
Sunset.
– Hipparchus.
Time of *autumnal equinox* (S_3 page 130).
Used by Hipparchus for determining the length of the tropical year.
Czwalina 1958, page 294 / Fotheringham 1918 / Petersen and Schmidt 1957, page 84 / A. Rome 1937, page 216.
35 **159 B.C. Sept 27** [Man 1, 134].
III Calippus 20 Epag. 1 [III, 1; Hei 1, 195].
Sunrise.
– Hipparchus.
Time of *autumnal equinox* (S_4 page 130).
See **34**.
Cf. the references to **34**.
36 **158 B.C. Sept 27** [Man 1, 134].
III Calippus 21 Epag. 1 [III, 1; Hei 1, 195].
Noon.
– Hipparchus.
Time of *autumnal equinox* (S_5 page 130).
See **34**.
Cf. the references to **34**.
37 **147 B.C. Sept 26/27** [Man 1, 134].
III Calippus 32 Epag. 3 [III, 1; Hei 1, 195].
Midnight.
Alexandria – Hipparchus.
Time of *autumnal equinox* (S_6 page 130).
See **34**; – also used with **89** by Ptolemy for checking the Hipparchian value of the tropical year (page 131 f.).
Cf. the references to **34**.

38 **146 B.C. March 24** [Man 1, 135].
III Calippus 32 Mechir 27 [III, 1; Hei 1, 196].
Morning.
Alexandria – Observed by Hipparchus with a ring (krikos).
Time of *vernal equinox* (S_7 page 130).
See **34**; – also used with **91** by Ptolemy for checking the Hipparchian value of the tropical year (page 131 f.).
Cf. the references to **34**; in particular Fotheringham 1918 b, page 416.

39 **146 B.C. April 21** [Man 1, 137 f.].
III Calippus 32 [III, 1; Hei 1, 199].
No time of the day recorded.
Alexandria – Hipparchus.
(Total) *lunar eclipse.*
Used with **43** by Hipparchus to derive a position of α Virginis 6½° west of the autumnal equinox.
Fotheringham 1918, page 417 f. / Ginzel 1911, page 540 / Rome 1937, page 216; 1938; and 1943, page 148 ff. / Tannery 1893, page 150.

40 **146 B.C. Sept 27** [Man 1, 134].
III Calippus 33 Epag. 4 [III, 1; Hei 1, 196].
Morning.
Alexandria – Hipparchus.
Time of *autumnal equinox* (S_8 page 130).
See **34.**
Cf. references to **34.**

41 **143 B.C. Sept 26** [Man 1, 134].
III Calippus 36 Epag. 4 [III, 1, Hei 1, 196]
Evening.
Alexandria – Hipparchus.
Time of *autumnal equinox* (S_9 page 130).
See **34.**
Cf. references to **34.**

42 **141 B.C. Jan 27** [Man 1, 351].
III Calippus 37 (Nabonassar 607) Tybi 2/3 [VI, 5; Hei 1, 478].
Beginning of the 5th hour [of the night].
Rhodes – Hipparchus
Beginning of partial *lunar eclipse*, 3 digits southern.
Used with **33** (page 227).
Boll 1909, col. 2358 / Ginzel 1899, page 234 / Newcomb, page 39 / Zech 1851, page 15.

43 **135 B.C. March 31** [Man 1, 137].
III Calippus 43 [III, 1; Hei 1, 199].
(Neither date nor time recorded).
– Hipparchus.
(Total) *lunar eclipse.*
Used with **39** by Hipparchus to derive a position of α Virginis 5¼° west of the autumnal equinox.
Fotheringham 1918, page 417 f. / Ginzel 1911, page 540 / A. Rome 1938, and 1943, page 148 ff. (quoting Theon of Alexandria) / Tannery 1893, page 150.

44 **135 B.C.** [Man 1, 145].
III Calippus 43 (End of the year) [III, 1; Hei 1, 207].
No time of the day recorded.
– Hipparchus.

Time of *Summer solstice.*
Used with **16** by Hipparchus for determining the length of the tropical year.

45 **135 B.C. March 23** [Man 1, 135].
III Calippus 43 Mechir 29 [III, 1; Hei 1, 196].
(Just) after midnight.
Hipparchus.
Time of *vernal equinox* (S_{10} page 130).
See **34** (cf. **43**).
References to **34**/Rome 1937, page 216.

46 **129/128 B.C.** [Man 2, 14].
III Calippus 50 [VII, 2; Hei 2, 15].
Neither date nor time of the day recorded.
Hipparchus.
Longitude of α Leonis = 119°;50.
Used with **83** for determining the rate of precession (page 245).
Rome 1937, page 216.

47 **128 B.C. March 23** [Man 1, 135].
III Calippus 50 Phamenoth 1 [III, 1; Hei 1, 196].
Sunset.
Rhodes? – Hipparchus.
Time of *vernal equinox* (S_{11} page 130).
See **34.**
Cf. references to **34.**

48 **128 B.C. Aug 5** [Man 1, 266].
III Calippus 51 (Nabonassar 619) Epiphi 16 [V, 3; Hei 1, 363].
When $\frac{2}{3}$ of the first hour of the day had passed.
Rhodes – Hipparchus.
Longitude 128°;35 *of* the *Sun and* 42°;20 of the *Moon.*
Used for verifying the evection (page 184).

49 **127 B.C. May 2** [Man 1, 271].
Alexander 197 (Nabonasser 620) Pharmuti 11 [V, 5; Hei 1, 369].
Beginning of the second hour [of the day].
Rhodes – Hipparchus.
Longitude 37°;45 *of the Sun and* 351°;40 of the *Moon.*
Used for testing the second lunar model by octant observations and demonstrating the prosneusis (page 189 ff.).

50 **127 B.C. July 7** [Man 1, 274].
Alexander 197 (Nabonassar 620) Payni 17 [V, 5; Hei 1, 374].
At $9\frac{1}{3}$ civil hours after sunrise.
Rhodes – Hipparchus.
Longitude 100°;54 *of the Sun and* 149° of the *Moon.*
Used for veryfying the prosneusis.

51 **A.D. 92 Nov 29** [Man 2, 23].
Domitian 12 Metros 7 (Nabonassar 840 Tybi 2/3) [VII, 3; Hei 2, 27].
Four civil hours before midnight.
Bithynia – Agrippa.
Occultation of the Pleiades by the Moon.
Used with **14** (page 247).
Fotheringham and Longbottom 1915.

52 **A.D. 98 Jan 11** [Man 2, 26].
Trajan 1 (Nabonassar 845) Mechir 15/16 [VII, 3; Hei 2, 30].
When the 10th hour of the night was complete.

Rome – Menelaüs.
Occultation of α Virginis by the Moon.
Used for investigating the possible influence of precession on latitude (page 247).
Fotheringham and Longbottom 1915, page 384 and 388.

53 **A.D. 98 Jan 14** [Man 2, 28].
Trajan 1 (Nabonassar 845) Mechir 18/19 [VII, 3; Hei 2, 33].
Towards the end of the 11th hour [of the night].
Rome – Menelaus.
Occultation of β Scorpii by the Moon.
See **52** (page 247).
Fotheringham and Longbottom 1915, page 384 and 390 f.

54 **A.D. 125 April 5** [Man 1, 239].
Hadrian 9 (Nabonassar 871) Pachon 17/18 [IV, 9; Hei 1, 329].
At $3\frac{3}{5}$ equinoctial hours before midnight.
Alexandria – Perhaps Ptolemy.
Beginning of partial *lunar eclipse* 2 digits southern (E_{12} page 170).
Used with **7** (page 181).
Boll 1909, col. 2361 / Ginzel 1899, page 234 / Newcomb, page 40 / Zech 1851, page 15.

55 **A.D. 127 March 26** [Man 2, 228].
Hadrian 11 Pachon 7/8 [XI, 5; Hei 2, 392].
Evening.
Alexandria – Ptolemy with the astrolabon.
Opposition of Saturn at a longitude of 181°;13 (P_1 page 273).
Used with **65** and **74** for determining the eccentricity and apogee (first approximation) of Saturn (page 273 ff.).
Czwalina 1958, page 296 ff.

56 **A.D. 127 Oct 12** [Man 2, 157].
Hadrian 12 Athyr 21/22 [X, 1; Hei 2, 297].
Morning.
– Theon of Smyrna.
Position of Venus relative to β Virginis giving a maximum western elongation of 47°;32 at a longitude of 150°;20 (V_3 page 299).
Used with **77** for determining the apogee of Venus (page 300).
Czwalina 1959, page 3 ff. / Wilson 1972, page 213.

57 **A.D. 129 May 20** [Man 2, 158].
Hadrian 13 Epiphi 2/3 [X, 2; Hei 2, 299].
Morning.
– Theon of Smyrna.
Position of Venus relative to α and 38 Arietis giving a maximum western elongation of 44°;48 at a longitude of 10°;36 (V_4 page 299).
Used with **76** for determining the epicycle radius and eccentricity of Venus, and for arguing that the apogee follows the fixed stars (page 307 f.).
Czwalina 1959, page 3, 6 / Wilson 1972, page 215.

58 **A.D. 130 July 4** [Man 2, 140].
Hadrian 14 Mesore 18 [IX, 9; Hei 2, 275].
Evening.
– Theon of Smyrna.
Position of Mercury 3°;50 east of α Leonis with a maximum eastern elongation of 26°;15 at a longitude of 126°;20 (M_8 page 310).
Used with **87** for determining the equant centre of Mercury (page 318).
Czwalina 1959 / Wilson 1972, page 232.

59 **A.D. 130 Dec 15** [Man 2, 176].
Hadrian 15 Tybi 26/27 [X, 7; Hei 2, 322].
One equinoctial hour after midnight.
Alexandria – Ptolemy with the astrolabon.
Opposition of Mars at a longitude of 81°.
Used with **70** and **85** for determining the eccentricity and apogee (first approximation) of Mars.
Czwalina 1958, page 300.

60 **A.D. 132 Feb 2** [Man 2, 130].
Hadrian 16 Phamenoth 16/17 [IX, 7; Hei 2, 262].
Evening.
Alexandria – Ptolemy with the astrolabon.
Position of Mercury relative to α Tauri giving a maximum eastern elongation of 21°;15 at a longitude of 331° (M_9 page 310).
Used with **67** for determining the apsidal line of Mercury (page 311), and with **94** for demonstrating one of the perigees (page 314, 321).
Czwalina 1959 / Wilson 1972, page 227.

61 **A.D. 132 March 8** [Man 2, 156].
Hadrian 16 Pharmuti 21/22 [X, 1; Hei 2, 296].
Evening.
– Theon of Smyrna.
Position of Venus relative to the Pleiades giving a maximum eastern elongation of 47°;15 at a longitude of 31°;30 (V_5 page 299).
Used with **93** for determining the apogee of Venus (page 300).
Czwalina 1959, page 3, 7 / Wilson 1972, page 213.

62 **A.D. 132 Sept 25** [Man 1, 184].
Hadrian 17 (Nabonassar 879) Athyr 7 [III, 7; Hei 1, 256].
Two equinoctional hours after noon.
Alexandria – Ptolemy.
Time of *autumnal equinox* (S_{12} page 130).
Used for determining the radix of the solar theory (page 151 ff.).
Fotheringham 1918, page 419 / A. Rome 1937, p. 216 / Petersen and Schmidt 1967, page 84.

63 **A.D. 133 May 6** [Man 1, 228].
Hadrian 17 Payni 20/21 [IV, 6; Hei 1, 314].
$\frac{3}{4}$ equinoctial hours before midnight.
Alexandria – Ptolemy.
Maximum of total *lunar eclipse* (E_{13} page 170).
Used with **69** and **73** for checking the epicycle radius of the Moon (page 177).
Boll 1909, col. 2361 / Ginzel 1899, page 234 / Newcomb, page 40 / Zech 1815, page 15.

64 **A.D. 133 May 17** [Man 2, 204].
Hadrian 17 Epiphi 1/2 [XI, 1; Hei 2, 360].
One hour before midnight.
Alexandria – Ptolemy with the astrolabon.
Opposition of Jupiter at a longitude of 233°;11.
Used with **75** and **78** for determining the eccentricity and apogee of Jupiter.
Czwalina 1958, page 299.

65 **A.D. 133 June 3** [Man 2, 228].
Hadrian 17 Epiphi 18 [XI, 5; Hei 2, 392].
Four hours after noon (found by interpolation between two observations).
Alexandria – Ptolemy with the astrolabon.
Opposition of Saturn at a longitude of 249°;40 (P_2 page 273).

See **55** and **74** (page 273 ff.).
Czwalina 1958, page 296.

66 **A.D. 134 Feb 18** [Man 2, 161].
Hadrian 18 Pharmuti 2/3 [X, 3; Hei 2, 303].
Morning.
Alexandria – Ptolemy.
Position of Venus relative to α Scorpii giving a maximum western elongation of 43°;35 at a longitude at 281°;55 (V_6 page 299).
Used with **90** for determining the equant centre of Venus (page 304).
Czwalina 1959, page 3, 8 / Wilson 1972, page 219.

67 **A.D. 134 June 4** [Man 2, 131].
Hadrian 18 Epiphi 18/19 [IX, 7; Hei 2, 262].
Early Morning.
Alexandria – Ptolemy with the astrolabon.
Position of Mercury relative to α Tauri giving a maximum western elongation of 21°;15 at a longitude of 48°;45 (M_{10} page 310).
Used with **60**; and with **79** for demonstrating one of the perigees of Mercury (page 314 and 321).
Czwalina 1959 / Wilson 1972, page 227.

68 **A.D. 134 Oct 3** [Man 2, 137].
Hadrian 19 Athyr 14/15 [IX, 8; Hei 2, 270].
Morning.
Alexandria – Ptolemy with the astrolabon.
Position of Mercury relative to α Leonis giving a maximum western elongation of 19°;3 at a longitude of 170°;12 (M_{11} page 310).
Used with **71** for determining the apogee (page 311) and the eccentricity (page 313) of Mercury.
Czwalina 1959 / Wilson 1972, page 229.

69 **A.D. 134 Oct 20** [Man 1, 228].
Hadrian 19 Choiak 2/3 [IV, 6; Hei 1, 314].
One equinoctional hour before midnight.
Alexandria – Ptolemy.
Maximum of a partial *lunar eclipse*, 10 digits southern.
Used with **2** and **6** (page 180), and with **63** and **73** (page 177).
Boll 1909, col. 2361 / Ginzel 1899, page 234 / Newcomb, page 40 / Zech 1851, page 15.

70 **A.D. 135 Feb 21** [Man 2, 176].
Hadrian 19 Pharmuti 6/7 [X, 7; Hei 2, 322].
Three equinoctial hours before midnight.
Alexandria – Ptolemy with the astrolabon.
Opposition of Mars at a longitude of 148°;50.
Used with **59** and **85**.
Czwalina 1958, page 300.

71 **A.D. 135 April 5** [Man 2, 137].
Hadrian 19 Pachon 19 [IX, 8; Hei 2, 270].
Evening.
Alexandria – Ptolemy with the astrolabon.
Position of Mercury relative to α Tauri giving a maximum eastern elongation of 23°;15 at a longitude of 34°;20 (M_{12} page 310).
Used with **68** (page 311) and with **60** and **94** for showing that there is no perigee opposite to the apogee of Mercury (page 314).
Czwalina 1959 / Wilson 1972, page 229.

72 **A.D. 135 Oct 1** [Man 1, 299].

Hadrian 20 Nabonassar 882 Athyr 13 [V, 13; Hei 1, 408].
At $5\frac{5}{6}$ equinoctial hours after noon.
Alexandria – Ptolemy with the parallactic instrument.
Zenith distance of the Moon = 50°;55.
Used for determining the parallax of the Moon (page 205).

73 **A.D. 136 March 6** [Man 1, 228].
Hadrian 20 Pharmuti 19/20 [IV, 6; Hei 1, 315]
Four equinoctial hours after midnight.
Alexandria – Ptolemy.
Maximum of partial *lunar eclipse*, 6 digits northern (E_{15} page 170).
Used with **63** and **69** (page 177).
Boll 1909, col. 2361 / Ginzel 1899, page 234 / Newcomb, page 40 / Zech 1851, page 16.

74 **A.D. 136 July 8** [Man 2, 228].
Hadrian 20 Mesore 24 [XI, 5; Hei 2, 393].
Noon (found by interpolation between two observations).
Alexandria – Ptolemy with the astrolabon.
Opposition of Saturn at a longitude of 284°;14 (P_3 page 273).
Used with **29** (page 289 f.), and with **55** and **65** (page 273 ff.).
Czwalina 1958, page 296.

75 **A.D. 136 Aug 21** [Man 2, 204].
Hadrian 21 Phaophi 13/14 [XI, 1; Hei 2, 360].
Two hours before midnight.
Alexandria – Ptolemy with the astrolabon.
Opposition of Jupiter at a longitude of 337°;54.
Used with **64** and **78.**
Czwalina 1958, page 299.

76 **A.D. 136 Nov 18** [Man 2, 159].
Hadrian 21 Tybi 2/3 [X, 2; Hei 2, 300].
Evening.
Alexandria – Ptolemy.
Position of Venus relative to α, ν, β Capricorni giving a maximum eastern elongation of 47°;20 at a longitude of 282°;50 (V_7 page 299).
Used with **57** (page 301).
Czwalina 1959, pp. 4, 8 / Wilson 1972, page 215.

77 **A.D. 136 Dec 25** [Man 2, 158].
Hadrian 21, Mechir 9/10 [X, 1; Hei 2, 298].
Evening.
Alexandria – Ptolemy.
Position of Venus relative to φ Aquarii giving a maximum eastern elongation of 47°;32 at a longitude of 319°;36 (V_8 page 299).
Used with **56** (page 300).
Czwalina 1959, pp. 4, 9 / Wilson 1972, page 213.

78 **A.D. 137 Oct 8** [Man 2, 204].
Antoninus 1 Athyr 20/21 [XI, 1; Hei 2, 360].
Five hours after midnight.
Alexandria – Ptolemy with the astrolabon.
Opposition of Jupiter at a longitude of 14°;23.
Used with **64** and **75** (cf. page 307), and with **88.**
Czwalina 1958, page 299.

79 **A.D. 138 June 4** [Man 2, 131].
Antoninus 1 Epiphi 20/21 [IX, 7; Hei 2, 263].

Evening.
Alexandria – Ptolemy with the astrolabon.
Position of Mercury relative to α Leonis giving a maximum eastern elongation of 26°;30 at a longitude of 97° (M_{13} page 310).
Used with **67**, and with **94** for determining the apsidal line of Mercury (page 311).
Czwalina 1959.

80 **A.D. 138 Dec 16** [Man 2, 164].
Antonius 2 Tybi 29/30 [X, 4; Hei 2, 306].
Early Morning (4¾ equinoctial hours after midnight).
Alexandria – Ptolemy with the astrolabon.
Position of Venus at a longitude of 216°;30 and a latitude of +2°;40 (V_9 page 299).
Used with **18** (page 307 f.).
Czwalina 1959, pp. 4, 9.

81 **A.D. 138 Dec 22** [Man 2, 243].
Antoninus 2 Mechir 6/7 [XI, 6; Hei 2, 414].
Four equinoctial hours before midnight.
Position of Saturn relative to α Tauri at a longitude of 309°;4 (P_4 page 273).
Used with **74** (page 285 ff., cf. page 307).

82 **A.D. 139 Feb 9** [Man 1, 265].
Antoninus 2 (Nabonassar 885) Phamenoth 25 [V, 3; Hei 1, 362].
5¼ equinoctial hours before noon.
Alexandria – Ptolemy with the astrolabon.
Longitude of the Sun = 318°;50 *and* of the *Moon* = 219°;40.
Used for demonstrating the evection (page 183).

83 **A.D. 139 Feb 23** [Man 2, 13].
Antoninus 2 Pharmuti 9 [VII, 2; Hei 2, 14].
Sunset.
Alexandria – Ptolemy with the astrolabon.
Longitude of α *Leonis* = 122°;30 (page 240 ff.).
Used with **46** (page 245).
Manitius 1905, page 406.

84 **A.D. 139 May 17** [Man 2, 147].
Antoninus 2 (Nabonassar 886) Epiphi 2/3 [IX, 10; Hei 2, 283].
4½ equinoctial hours before midnight.
Alexandria – Ptolemy with the astrolabon.
Position of Mercury relative to α Leonis at a longitude of 77°;30 (M_{14} page 310).
Used with **20** (page 324); cf. page 307).

85 **A.D. 139 May 27** [Man 2, 176].
Antoninus 2 Epiphi 12/13 [X, 7; Hei 2, 322].
Two equinoctial hours before midnight.
Alexandria – Ptolemy with the astrolabon.
Opposition of Mars at a longitude of 242°;34.
Used with **59** and **70** (cf. page 307).
Czwalina 1958, page 301.

86 **A.D. 139 May 30** [Man 2, 195].
Antoninus 2 Epiphi 15/16 [X, 8; Hei 2, 347].
Three equinoctial hours before midnight.
Alexandria – Ptolemy with the astrolabon.
Position of Mars relative to α Virginis at a longitude of 241°;36.
Used with **85** for determining the epicycle radius of Mars.

87 **A.D. 139 July 5** [Man 2, 141].

Antoninus 2 Mesore 21 [IX, 9; Hei 2, 275].
Dawn.
Alexandria – Ptolemy with the astrolabon.
Position of Mercury relative to α Tauri giving a maximum western elongation of 20°;15 at a longitude of 80°;5 (M_{15} page 310).
Used with **58** (page 318 ff.).
Czwalina 1959 / Wilson 1972, page 232. / Already Copernicus (*De Rev.* V, 27) put this observation 3 days earlier.

88 **A.D. 139 July 11** [Man 2, 220].
Antoninus 2 Mesore 26/27 [XI, 2; Hei 2, 382].
Five equinoctial hours after midnight.
Alexandria – Ptolemy with the dioptra.
Position of Jupiter relative to α Tauri and the Moon at a longitude of 75°;45.
Used with **78** for determining the epicycle radius of Jupiter.

89 **A.D. 139 Sept 26** [Man 1, 142].
Antoninus 3 (Alexander 463) Athyr 9 [III, 1; Hei 1, 204].
About one hour after sunrise.
Alexandria – Ptolemy.
Time of *autumnal equinox* (S_{13} page 130).
Used with **37** (page 132), and with **91** and **92** for determining the eccentricity and apogee of the Sun (page 145).
Cf. references to **34.**

90 **A.D. 140 Feb 18** [Man 2, 162].
Antoninus 3 Pharmuti 4/5 [X, 3; Hei 2, 303].
Evening.
Alexandria – Ptolemy.
Position of Venus relative to α Tauri giving a maximum eastern elongation of 48°;20 at a longitude of 13°;50 (V_{10} page 299).
Used with **66** (page 304).
Czwalina 1959, pp. 4, 10 / Wilson 1972, page 219.

91 **A.D. 140 March 22** [Man 1, 143].
(Antoninus 3) Alexander 463 Pachon 7 [III, 1; Hei 1, 205].
One hour after noon.
Alexandria – Ptolemy.
Time of *vernal equinox* (S_{14} page 130).
Used with **38** (page 132), and with **89** and **92** (page 145).
Cf. References to **34.**

92 **A.D. 140 June 25** [Man 1, 144].
Antoninus 3 (Alexander 463) Mesore 11/12 [III, 1; Hei 1, 206].
About two hours after midnight.
Alexandria – Ptolemy.
Time of *Summer solstice* (S_{15} page 130).
Used with **8** and **16** (page 132), and with **89** and **91** (page 145).
Cf. references to **34.**

93 **A.D. 140 July 30** [Man 2, 156].
Antoninus 4 Thoth 11/12 [X, 1; Hei 2, 297].
Morning.
Alexandria – Ptolemy.
Position of Venus relative to ζ Geminorum giving a maximum western elongation of 47°;15 at a longitude of 78°;30 (V_{11} page 299).

Used with **61** (page 300); cf. page 307.
Cf. note 5, page 299 / Czwalina 1959 / Wilson 1972, page 213.

94 **A.D. 141 Feb 2** [Man 2, 131].
Antoninus 4 Phamenoth 18/19 [IX, 7; Hei 2, 263].
Early Morning.
Alexandria – Ptolemy with the astrolabon.
Position of Mercury relative to α Scorpii giving a maximum western elongation of 26°;30 at a longitude of 283°;30 (M_{16} page 310).
Used with **60** (page 314, 321), and with **79** (page 311).
Cf. page 307 / Wilson 1972, page 227.

Numerical Parameters

Italics indicate values implicit in Ptolemaic astronomy, but not actually quoted in the Almagest.
Brackets contain modern values calculated for Ptolemy's time.

General Constants

Obliquity of the ecliptic	$\varepsilon = 23°;51,20$	[23°;40, 50]
Rate of precession	$p = 1°$ per century	
	$= \mathit{36''}$ *per year*	[49''.86]
Latitude of Alexandria	$\varphi = 30°;58$	[31°;13]
Standard epoch	$t_0 =$ Nabonassar 1, Thoth 1, Noon	
	= *747 B.C., Feb. 26, Noon*	

The Sun

Mean motion in longitude	$\omega_{\odot} = 0°;59,08,17,13,12,31$ per day	
Mean periods		
One tropical year	$T_t = 365 + \frac{1}{4} - \frac{1}{300}$ days	
	$= 365^d;14,18$	
	$= \mathit{365^d5^h55^m12^s}$	
	$= \mathit{365^d.24666}$	[$365^d.24220$]
One sidereal year	$T_f = \mathit{365^d.25682}$	[$365^d.25636$]
Radix of mean motion	$\lambda_m(t_0) = 330°;45$	
Geometrical parameters	$R = 60^p$	
	$e = 2^p;30$	
	$\max q = 2°;23$	
	$\lambda_a = 65°;30 =$ constant	

The Moon

Mean daily motions in

longitude	ω_t	$= 13°;10,34,58,33,30,30$
elongation	ω_s	$= 12°;11,26,41,20,17,59$
anomaly	ω_a	$= 13°;03,53,56,17,51,59$
'latitude'	ω_d	$= 13°;13,45,39,48,56,37$

Mean periods

tropical	T_t	$= 27^d;19,37,02,45$	
		$= 27^d07^h43^m07^s.3$	$[27^d07^h43^m04^s.7]$
		$= 27^d.321612$	$[27^d.321582]$
synodic	T_s	$= 29^d;31,50,08,20$	
		$= 29^d12^h44^m03^s.3$	$[29^d12^h44^m02^s.8]$
		$= 29^d.530594$	$[29^d.530588]$
anomalistic	T_a	$= 27^d;33,16,27,22$	
		$= 27^d13^h18^m34^s.9$	$[27^d13^h18^m33^s.1]$
		$= 27^d.554571$	$[27^d.554550]$
draconitic	T_d	$= 27^d;12,43,59,21$	
		$= 27^d05^h05^m35^s.7$	$[27^d05^h05^m35^s.8]$
		$= 27^d.212219$	$[27^d.212220]$

Radices of mean motion

in longitude	$\lambda_m(t_0) = 41°;22$
in elongation	$\lambda_{m☽}(t_0) - \lambda_{m\odot}(t_0) = 70°;37$
in anomaly	$a(t_0) = 268°;49$
in 'latitude'	$\lambda_\beta(t_0) = 354°;15$

Geometrical parameters

$R = 49^p;41$	or	60^p	inclination i	$= 5°;00 =$ constant	
$r = 5^p;15$	or	$6^p;20$	max parallax	$1°;07$	$[0°;57]$
$e = 10^p;19$	or	$12^p;29$	mean distance	59^r	$[60.4^r]$
$\max p = 5°;01$	to	$7°;40$			

Mercury

Mean motion	
in longitude	$\omega_t = 0°;59,08,17,13,12,31$ per day $= \omega_\odot$
in anomaly	$\omega_a = 3°;06,24,06,59,35,50$ per day
Mean periods	
tropical	$T_t =$ one tropical year
synodic ('anomalistic')	$T_a = \mathit{115^d.88}$ [$115^d.88$]
	$= \mathit{0.317}$ tropical year
Radices of mean motion	
in longitude	$\lambda_m(t_0) = 330°;45$
in anomaly	$a_m(t_0) = 21°;55$

Geometrical parameters

$R = 60^p$	$\lambda_a - \lambda_n = 90°$
$r = 22^p;30$	$i_m = 0°$ to $-0;45$
$e = 3^p;00$	$j_m = 6°;15$
$\max q = 3°;02$	$k_m = 7°;00$
$\max p = 23°;53$	
$\lambda_a(t_0) = 181°;10$	

Arc of vision $h = 10°$

Venus

Mean motion

in longitude	$\omega_t = 0°;59,08,17,13,12,31$ per day $= \omega_{\odot}$
in anomaly	$\omega_a = 0°;36,59,25,53,11,28$ per day

Mean periods

tropical	$T_t =$ one tropical year
synodic ('anomalistic')	$T_a =$ *583^d.98* [583^d.92]
	$=$ *1.599* tropical years

Radices of mean motion

in longitude	$\lambda_m(t_0) = 330°;45$
in anomaly	$a_m(t_0) = 71°;07$

Geometrical parameters

$R = 60^p$	$\lambda_a - \lambda_n = 90°$
$r = 43^p;10$	$i_m = 0°$ to $+0°;10$
$e = 1^p;15$	$j_m = 2°;30$
$\max q = 2°;24$	$k_m = 3°;30$
$\max p = 47°;15$	
$\lambda_a(t_0) = 46°;10$	

Arc of vision $h = 5°$

Mars

Mean motion

in longitude	$\omega_t = 0°;31,26,36,53,51,33$ per day
in anomaly	$\omega_a = 0°;27,41,40,19,20,58$ per day

Mean periods

tropical	$T_t =$ *687ᵈ.04*	[686ᵈ.93]
	$=$ *1.88* tropical years	
synodic ('anomalistic')	$T_a =$ *779ᵈ.94*	[779ᵈ.94]
	$=$ *2.14* tropical years	

Radices of mean motion

in longitude	$\lambda_m(t_0) = 3°;32$
in anomaly	$a_m(t_0) = 327°;13$

Geometrical parameters

$R = 60^p$	$\lambda_a - \lambda_n \approx 85°;30$
$r = 39^p;30$	$i = 1°;00$
$e = 6^p;00$	$j_m = 2°;15$
$\max q = 11°;25$	
$\max p = 46°;59$	
$\lambda_a(t_0) = 106°;40$	

Arc of vision $h = 11\frac{1}{2}°$

Jupiter

Mean motion	
in longitude	$\omega_t = 0°;04,59,14,26,46,31$ per day
in anomaly	$\omega_a = 0°;54,09,02,46,26,00$ per day
Mean periods	
tropical	$T_t = \mathit{4332^d.38}$ [4331^d]
	$= \mathit{11\ .86}$ tropical years
synodic ('anomalistic')	$T_a = \mathit{398^d,88}$ [$398^d.88$]
	$= \mathit{1\ .09}$ tropical years
Radices of mean motion	
in longitude	$\lambda_m(t_0) = 184°;41$
in anomaly	$a_m(t_0) = 146°;04$

Geometrical parameters

$R = 60^p$	$\lambda_a - \lambda_n \approx 71°$
$r = 11^p;30$	$i = 1°;30$
$e = 2^p;45$	$j_m = 2°;30$
$\max q = 5°;15$	
$\max p = 11°;36$	
$\lambda_a(t_0) = 152°;09$	

Arc of vision $h = 10°$

Saturn

Mean motion

in longitude	$\omega_t = 0°;02,00,33,31,28,51$ per day
in anomaly	$\omega_a = 0°;57,07,43,41,43,40$ per day

Mean periods

tropical	$T_t =$ *10770*^{d}	[10747^d]
	= *29 .49* tropical years	
synodic ('anomalistic')	$T_a =$ *378*^{d}*.09*	[$378^d.09$]
	= *1 .035* tropical years	

Radices of mean motion

in longitude	$\lambda_m(t_0) = 296°;43$
in anomaly	$a_m(t_0) = 34°;02$

Geometrical parameters

$R = 60^p$	$\lambda_a - \lambda_n \approx 133°$
$r = 6^p;30$	$i = 2°;30$
$e = 3^p;25$	$j_m = 4°;30$
$\max q = 6°;31$	
$\max p = 6°;36$	
$\lambda_a(t_0) = 224°;10$	

Arc of vision $h = 11°$

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Supplementary Notes

Notes that are adapted in wording or substance from Toomer (1977) or Saliba (1975) are marked by the initials [GJT] or [GS]. Cited works are listed in the Supplementary Bibliography except for those marked by an asterisk, which will be found in Pedersen's bibliography.

p. 12 The most authoritative survey of Ptolemy's life and works is now Toomer (1975), to which Jones (2007) is a supplement.

Ptolemy's inscription at Canopus, which was dedicated in A.D. 146/147 (not 147/148), has been shown to have preceded the completion of the *Almagest*; see Hamilton, Swerdlow, & Toomer (1987). For an edition, translation, and further discussion see Jones (2005). The divergences between the inscription and the *Almagest* with respect to some of the parameters of the lunar and planetary models represent changes that Ptolemy made in the course of preparing and writing the *Almagest*, and he refers to them in IV, 9 [Hei 1, 326-328].

"Claudios" is not a Greek name, but a Hellenization of the Latin "Claudius": its importance is that it shows that its bearer had Roman citizenship. "Ptolemaios" certainly does not indicate that a man so named was born in a town called Ptolemais: it is common in all parts of Egypt. We do, however, find an explicit statement that Ptolemy came from Ptolemais Hermeiou, but not until the 14th century (Theodorus Meliteniotes). See Boll (1894)* 54-55. [GJT]

p. 13 Pedersen states as a fact that the Theon who "gave" Ptolemy some observations is Theon of Smyrna. The identification has indeed been suggested, and is chronologically possible, but "Theon" is such a common name, especially in Egypt, that it remains pure speculation. [GJT]

In both the *Handy Tables* and the (still later) *Planetary Hypotheses* the models for the planets' latitudinal motion are different from, and on the whole superior to, those of *Almagest* XIII; see Swerdlow (2005). The lunar model underlying the *Handy Tables* was not, in fact, different from that of the *Almagest*, though Ptolemy introduced some minor modifications in the *Planetary Hypotheses*.

p. 14 For a recent and balanced discussion of the relation of Indian to Greek astronomy see Plofker (2008) 61-120.

The section on the Arabic translations of the *Almagest* is a tissue of errors which would take too much space to disentangle here. Fortunately one can now refer to Kunitzsch's excellent book (1974), pp. 15-82 (on the name "Almagest" ibid. pp. 115-125). [GJT]

p. 15 The Leiden manuscript which is alleged to contain a version "by an anonymous scholar" of the year 827 is in fact the sole existing manuscript of the version of al-Ḥajjāj, whom Pedersen mentions just below as composing his version in 829-830; 827-828 is the correct date. [GJT]
The work referred to is al-Farghānī, Muh. b. Kathīr, *'uṣūl 'ilm al-nujūm*, published in al-Farghānī (1669). [GS]

p. 17 All the sections of the "Speculum astronomicum" referring to ancient and medieval astronomy have been edited by Cumont in Cumont & Boll (1904), 85-105. [GJT]

On the so-called "Almagestum parvum", which is an anonymous Latin compilation rather than a translation from the Arabic by Gerard of Cremona, see Lorch (1992).

p. 21 The first printing of the Greek *Almagest* was not prepared "from a (now lost) MS at Nuremberg." The manuscript Nuremberg Cent. V 8 2° (which still exists) *was* used for the text of Theon's commentary on the *Almagest*, which was published in the editio princeps of the Greek text. But the text of the *Almagest* proper was taken from a much later (16th century) manuscript. See Heiberg's edition of Ptolemy's *Opera Astronomica Minora*, pp. xxv, lxxiii-lxxiv. [GJT]

p. 25 Neither Pedersen nor Neugebauer, in his treatment of the *Almagest* in Neugebauer (1975), gave much attention to Ptolemy's observation reports and the question of their authenticity. The topic has since returned to prominence in studies of Ptolemy's work. In a series of publications, and most prominently in his book *The Crime of Claudius Ptolemy* (1977), R. R. Newton revived Delambre's charges against Ptolemy, maintaining that he systematically fabricated his observation reports, even including those he attributed to his predecessors, on the basis of models and parameters taken over from earlier astronomers. Newton's arguments were of very unequal merit—his naïve calculations of the probability of genuineness of the observation reports invited especial ridicule—but several among them make a convincing case that Ptolemy frequently constructed or tampered with observation reports or skewed calculations in order to obtain preconceived parameters. Whether these parameters were the results of his own research,

following different methods from the ones presented in the *Almagest*, or taken over from predecessors is in general difficult to establish because so little evidence survives for observational technique and for the development of kinematic astronomy between Hipparchus and Ptolemy. A selection of recent publications dealing with aspects of this subject includes Swerdlow (1989), Graßhoff (1990), Britton (1992), Duke (2005a), Jones & Duke (2005), and Jones (2006).

p. 35 See Ibn al-Haytham, *Al-Shukūk ʻalā Batlamyūs*, published in Ibn al-Haytham (1971). [GS]

p. 36 Pedersen infers from Ptolemy's argument that sundials would not show the correct time if the heavens did not move like a sphere that he is thinking of the hemispherical bowl sundial, called πόλος. If true, this would be very interesting historically. However, *all* sundials presuppose motion on a sphere, so the inference is invalid. [GJT] The sundials known to the Babylonians, and almost certainly also that known to Herodotus, were in fact planar.

p. 42 Cleomedes' date is uncertain (Bowen & Todd (2004) 2), and his book may have been written later than the *Almagest*. In any case there is no particular reason why Ptolemy should have cited Cleomedes for the phenomenon of atmospheric refraction, which was surely not his discovery. Pliny the Elder, *Naturalis historia* II 56 reports that Hipparchus knew of (indeed may have himself observed, see Toomer (1980) 105) a lunar eclipse during which both Sun and Moon were above the horizon at mid-eclipse.

p. 48 Although Ptolemy frequently assumes theorems proved in Euclid's *Elements*, he nowhere refers explicitly to either the author or the book.

p. 51 29^{d};31,50,8,20 is in fact a Babylonian value for the mean synodic month (see supplementary note to p. 162). If, as seems probable, Ptolemy's report that Hipparchus endorsed this value is accurate, Hipparchus certainly was familiar with sexagesimal notation; it does not occur, however, in his one surviving writing (the commentary on Aratus' and Eudoxus' *Phaenomena*) or in other testimonia to Hipparchus' astronomical works.

Hypsikles' *Anaphorikos* is said to have been written "about 200 B.C." This would make Hypsikles a contemporary of Apollonius of Perge, whereas he himself says ("Euclid", *Elements* XIV, Preface) that his *father* had criticized a work of Apollonius. The biographical data available on Hypsikles (see e.g. Fraser (1972) II p. 612) show that the *Anaphorikos* can hardly predate 150 B.C. [GJT]

The modifications of the Babylonian sexagesimal notation that Pedersen describes were certainly not due to Ptolemy, since they are attested in

numerous Greek astronomical papyri from the first century A.D. onwards; see for example the first-century papyrus *P. Oxy. astron.* 4136 in Jones (1999) II 14-15.

p. 52 For the forms of the Greek zero symbol, most commonly a small circle or dot below a horizontal stroke, see Jones (1999) I 62-63. The symbol probably has nothing to do with the letter omicron, which carries the numerical value 70 in the Greek notation.

p. 56 Toomer (1973) argued, on the basis of a hypothetical reconstruction of the steps by which Hipparchus arrived at the ratios of lunar eccentricity that Ptolemy reports in *Almagest* IV 11, that he used a chord table related to later Indian sine tables, in which the standard circle's circumference was set at 21600 units and its diameter at 3438 units. Toomer subsequently repudiated this conclusion because his reconstruction was in part based on an incorrect version of Ptolemy's report (see Toomer (1984) 215 note 75), but it has recently been defended in Duke (2005b).

p. 60 As Tropfke (1928)* pp. 445-446 and Toomer (1973) p. 18 remark, a theorem substantially identical to Ptolemy's for determining the chord of half an angle whose chord is known is found in a short work on the construction of the regular heptagon preserved in Arabic under Archimedes' name. For this text and its history see Hogendijk (1984), who on pp. 212-213 concludes that the accuracy of the attribution cannot be determined. Toomer, pp. 20-23, combats Tropfke's opinion, cited above on p. 59, that Archimedes possessed the theorem for determining the chord of the sum or difference of two angles whose chords are known.

p. 61 Pedersen says that Ptolemy "realized" that trisection of an angle was not in general possible. This is no more than an inference: what Ptolemy says (Hei 1, 42, lines 18-20) is that the *chord* corresponding to an arc which is one-third of the arc of a given chord cannot be found by geometrical methods (διὰ τῶν γραμμῶν). He gives no justification for this statement. It is of course legitimate to reason that, if Crd θ could be found from given Crd 3 θ, then it would be simple to construct θ from given 3 θ, and, since the latter is impossible, therefore the former is also impossible; but we cannot say for certain that this is Ptolemy's own reasoning. An indirect proof of the impossibility of constructing Crd θ is in Neugebauer (1975) p. 24 note 3 (which is however incorrect as it stands). A more direct proof is: from the essential equivalence of chord function and sine function, and the identity $\sin 3\theta = -4\sin^3\theta + 3\sin\theta$, it follows that, given Crd 3 θ = *A*, finding Crd θ = *x* amounts to solving the equation $x^3 - \frac{3}{4}x + \frac{A}{4} = 0$. It is easily shown that, in general, this equation has no rational root, and hence *x*, or Crd θ, cannot be found by ruler and compass constructions. [GJT]

p. 65 On Ptolemy's plane and spherical trigonometry see now van Brummelen (2009) 68-93.

p. 69 Besides the observation report in the *Almagest*, Menelaus appears as a character in Plutarch's "Face in the Moon," and his relationship to the emperor Domitian is attested by two of his works which are extant in Arabic translation. For bibliographical details see Sezgin (1974) pp. 161-164.

p. 73 The history of transmission of Menelaus' theorem in Arabic texts is complex; see Sidoli (2006), who shows that the theorem in the edition of Menelaus by Abū Naṣr Manṣūr (which is published in Krause (1936)) actually derives from the *Almagest*, so that it is not established that Ptolemy depended on Menelaus.

p. 78 Ptolemy does not claim that "he has dealt with *all* the mathematical prerequisites of the *whole* study of astronomy and geography," but merely that he has "dealt with those aspects of heaven and earth which required, in outline, a preliminary mathematical discussion." [GJT]

p. 79 The Antikythera Mechanism, a fragmentarily preserved gearwork planetarium dating from the second or early first century B.C., is now recognized as having represented the Moon's motion according to an eccentric or epicyclic model; it likely also had similar devices for the Sun and planets. See Freeth et al. (2006) and Wright (2002).

p. 81 The claim that Ptolemy's was the first trigonometrical function tabulated in the history of Greek mathematics, besides contradicting what Pedersen writes on p. 56 adducing evidence respecting Hipparchus and Menelaus, is exceedingly improbable; it would imply that no astronomer before Ptolemy had produced a trigonometrically based anomaly table for a heavenly body.

p. 89 The explanation offered for the medieval term "diversitas diametri circuli brevis," that it reflects an epicycle of varying size, seems most improbable. The term is derived from the Arabic (it occurs in the Toledan Tables for the Moon, see Toomer (1968) p. 58). Toomer discusses its meaning and origin in Benjamin & Toomer (1971) p. 404 note 104. Pedersen claims that the term is applied to the planetary models, but the passage he quotes from the *Theorica Planetarum* refers only to the Moon, and this is apparently the only context in which it occurs in other authors. [GJT]

p. 95 Ptolemy says that the obliquity ε he found is approximately the same as the value assumed by Eratosthenes and Hipparchus, and that its ratio to half the meridian circle is approximately 11 to 83 (so that to two sexagesimal places ε = 23;51,20°). He does *not* say that either Eratosthenes or Hipparchus actually assumed either that ratio or the precise value 23;51,20°. For an argument that Eratosthenes derived from rather crude geographical data a

value equivalent (in whatever units he used) to $23\frac{6}{7}$, i.e. approximately 23;51,26°, see Jones (2002). Hipparchus may have used different values for ε at various times, certainly including the round figure 24°, but also perhaps one closer to Ptolemy's value (Goldstein (1983)) and, most accurately, one near 23;40° (Diller (1934)).

p. 101 Following a mistranslation by Manitius, Pedersen attributes to Ptolemy the statement that the Earth is "at present" inhabited only upon one half of the northern hemisphere. In context the expression καθʿ ἡμᾶς is spatial, not temporal: ἡ καθʾ ἡμᾶς οἰκουμένη means "our part of the inhabited world." [GJT]

p. 103 Pedersen solves a more general problem than Ptolemy, who only deals with the particular case when one is given the length of the *longest* day for a locality (hence δ = ε). Ptolemy also shows how to derive the length of the longest day M from a given latitude φ. His procedures for converting either way are equivalent to using the relation $-\cos\frac{M}{2} = \tan\varphi\tan\varepsilon$.

p. 107 Ptolemy's "Tanais" is certainly the Don, as is evident from his *Geography*, V 9, where he places the mouth of the Tanais at the city of the same name at the north end of the Sea of Azov ("Lake Maiotis"), with latitude $54\frac{2}{3}$°. [GJT]

p. 108 The numbers in Pedersen's table have been brought into agreement with the readings adopted by Toomer (1984). Some of the figures for the minutes are uncertain, since Ptolemy's calculations, especially for the more northerly latitudes, are apparently imprecise.

p. 109 Ptolemy's latitude for Alexandria is, to the nearest minute of arc, what one would derive from the "traditional" datum that a vertical gnomon at Alexandria casts a shadow $\frac{7}{5}$ its length at noon on an equinox. Whatever measurements he may have made using his meridan instruments, he was undoubtedly influenced by the traditional parameter.

Strabo say, not that the terms "amphiscian" etc. were *coined* by Posidonius, but that they were *used* by him; they may well be much older. [GJT]

For a translation of the theoretical sections of the *Geography* see Berggren & Jones (2000); Stückelberger et al. (2006) provide an excellent complete edition with German translation and reconstructions of the maps.

p. 111 The formulas for the calculation of rising-times which Pedersen develops are mathematically correct, but are not a good representation of Ptolemy's actual procedure. If one were to recompute the tables for rising-times using

them, together with Ptolemy's values for the latitude φ, one would find noticeable discrepancies from his results. The reason is that Ptolemy computes the rising-times not, as one would infer from Pedersen's presentation, from φ, but from the length of the longest day, M; and although φ and M are connected by $-\cos\frac{M}{2} = \tan\varphi\tan\varepsilon$, Ptolemy's values for φ are slightly rounded. [GJT]

p. 121 Pedersen's statements of the symmetrical properties of angles between the ecliptic and a vertical are wrong. He has telescoped four statements into two and combined them incorrectly. The correct formulation is:

A. If two *different* points of the ecliptic, H and J, are symmetric to a solstice and equidistant from the meridian,

$$z(H) = z(J)$$

$$\text{and } \psi(H) + \psi(J) = 180°$$

B. If H_1 and H_2 are two positions of the *same* point of the ecliptic, H, which are symmetric to the meridian,

$$z(H_1) = z(H_2)$$

$$\text{and } \psi(H_1) + \psi(H_2) = 2\gamma,$$

where γ is the angle between ecliptic and meridian when H is in the meridian.

Since four angles are formed by the intersection of the ecliptic and vertical great circles, it is important to note which one Ptolemy means by the angle between them. This is the angle that is simultaneously within the hemisphere bounded by the ecliptic and containing the north celestial pole and within the hemisphere bounded by the vertical and containing the westwards portion of the ecliptic. [GJT]

p. 124 According to Pedersen, the use of the Egyptian calendar (with unvarying years of 365 days) in the *Almagest* is one indication that in Egypt "Greek chronology and time reckoning had never succeeded in replacing the traditional Egyptian calendar." But in Ptolemy's time (cf. p. 126) the calendar in common civil use in Alexandria was the *modified* Egyptian calendar in which an extra day was intercalated every fourth year. Ptolemy used the Egyptian calendar, not because he "lived and worked in Egypt," but because it was convenient for tabulation purposes to have a year of unvarying length. It is clear from the *Almagest* and elsewhere that the unvarying Egyptian year was used by astronomers in many parts of the Greek world before Ptolemy.

Pedersen says that Ptolemy's "astronomical" system of measuring time (by equinoctial hours) is "essentially the same as we use today." There is an essential difference. The *Almagest* timing of an event is always with respect to "true noon," i.e. the time when the Sun is in the meridian on the day and place in question. We have days of uniform length, counted from a "mean noon" (or midnight). Hence the importance of the "equation of time" in the *Almagest*. [GJT]

p. 126 Ptolemy uses "double dates" regularly for civil times falling between sunset and sunrise, not only after midnight. In this he was following an established convention in Egypt (attested for example in numerous personal horoscopes preserved on papyri) that eliminated any ambiguity about whether the night in question preceded or followed the day having the same number. Pedersen's preferred translation is therefore incorrect; better would be (e.g.) "Mesore the 11th continuing into the 12th."

Julius Caesar's calendar reform, instituted in 45 B.C., applied to the *Roman* calendar. The analogous reform of the Egyptian calendar was instituted by Augustus after Egypt became a Roman province, probably in 26 B.C. In Egypt the Roman calendar was seldom used except in contexts involving Roman administration. The *Almagest* contains no instance of a date recorded according to either the Roman or the reformed Egyptian calendar.

Toomer (1984) p. 11 gives an accurate summary of the *Canon Basileion* in a form convenient for use with the *Almagest*. The original table is preserved in manuscripts of Ptolemy's *Handy Tables*, and in all copies it has been extended far beyond the period when Ptolemy himself lived.

p. 127 Fragments survive of a series of eight cuneiform tablets comprising comprehensive series of Babylonian records of lunar eclipses covering more than 330 years starting with an eclipse in 747 B.C. (February 6, just 20 days before Ptolemy's "era of Nabonassar"); see Steele (2000) pp. 432-433. This was probably the ultimate source of the Babylonian eclipse records known to Hipparchus and Ptolemy. Babylonian records of other kinds of astronomical observations, such as planetary phenomena, are first attested in the seventh century B.C., though Ptolemy cites none earlier than the third.

Strictly speaking, Ptolemy's citations of years reckoned from the accessions of Roman emperors (other than Augustus) are just instances of regnal years, not eras. His sole passing reference to the era of Augustus is in *Almagest* III 7, the calculation of the epoch position of the Sun's motion; in the *Canobic Inscription* all the epochs are given for the era of Augustus.

p. 128 The Babylonian lunisolar calendar had its intercalary months regulated by a cycle of 19 years comprising 235 lunar months starting about 500 B.C. This was almost certainly the inspiration for Meton's calendar. It was a Greek

innovation (whether by Meton himself is unclear) to fix also the number of days in a cycle and specify for all time, not only which years of the cycle should have 12 or 13 months, but also which months of the cycle should have 29 or 30 days.

So far as we know, the Callippic calendar, with years counted within 76-year Callippic periods, was *only* used in astronomical contexts; examples of Callippic dates have recently been found in Greco-Egyptian astronomical papyri running as late as the 6th Callippic period in the first century A.D.; see Jones (2000). A calendar structured according to the Metonic or Callippic cycles has been found inscribed on the Antikythera Mechanism (see supplementary note to p. 79, and Freeth et al. (2008)).

p. 131 The error in Hipparchus' value for the tropical year appears small, but applied to the roughly three centuries between his time and the oldest solar observation available to him (the solstice of Meton in 432 B.C.) or the roughly three centuries between his time and Ptolemy's, it amounts to about a day's error, which is significant. The causes of the error in Hipparchus' original determination are not recoverable, and certainly should not be merely attributed "partly to imperfect instruments, partly to the influence of atmospheric refraction," especially since we do not know what instruments Meton, Aristarchus, and Hipparchus used. From Ptolemy's account, it appears that Hipparchus used the equinox observations S_3-S_{11} only in his consideration of the short-term behavior of the tropical year and the question of whether it is a constant. To obtain a precise *value*, he compared summer solstice observations spanning as long an interval as possible, and apparently he most trusted the pairing of Aristarchus' solstice in 280 B.C. with a solstice he himself observed in 135 B.C. (not listed by Pedersen, but see *Almagest* III 1, Hei 1, 207). Ptolemy does not repeat the precise observed date and time of either of these solstices, but it is likely that Hipparchus' was fairly accurate while Aristarchus' was recorded for a time a large fraction of a day earlier than the actual moment of solstice. On the topic of Hipparchus' study of the tropical year and Ptolemy's reporting of it, see Jones (2005).

Ptolemy explicitly says (*Almagest* III 1; Hei 1, 203) that he used his meridian instruments, described in I 12, to determine the dates and times of equinoxes and solstices. Since these instruments are designed to make measurements of the Sun's altitude at noon only, the precise moment of equinox or solstice has to be determined by some kind of interpolation, and a solstice in particular would require measurements for several days before and after the event because of the extremely small daily change in the noon altitude around solstice. Whether any reasonable interpolation method would result in precision to the nearest hour, such as Ptolemy claims, is open to doubt, but this has more bearing on the genuineness of Ptolemy's reports than on the nature of the instrument.

The equatorial rings at Alexandria were only usable for equinoxes, though they had the specious advantage of appearing to show the precise moment when the Sun crosses the equatorial plane so long is this occurs during the daytime. Ptolemy cites Hipparchus for an observation on a bronze ring at Alexandria of the vernal equinox in 146 B.C., according to which it took place an hour before noon whereas Hipparchus (in Rhodes?) had calculated the time as sunrise. In Ptolemy's own time, by his report, there were two such rings in the Palaestra at Alexandria, one of which was comparatively old. He disparages them as worthless for accurate observation because of faultly alignment, but does not otherwise describe them. It is Theon, in the passage to which Pedersen refers, who speaks of his own experience with observing the shadow cast upon itself by a ring of "not less than two cubits," though it is not clear whether this was one of the publicly mounted rings extant in Ptolemy's time or even just the equatorial ring of an armillary.

p. 132 Ptolemy does not compare his summer solstice observation S_{15} with Aristarchus' (S_2), although he does count the number of intervening years between Meton's (S_1) and S_{15} in stages of which one is the *year* of Aristarchus' observation.

Pedersen does not discuss the reason why Ptolemy should have confirmed a value of the tropical year that is too high. Hipparchus' determination of this value was based on a comparison of his own solstice observations with a very small number of older ones—perhaps just the two ascribed to Meton and Aristarchus—and it is plausible that there would have been either instrumental inaccuracies in these early observations or uncertainties in the transmission of the dates. Ptolemy is working with a wider variety of observations (equinoxes as well as solstices), presumably a larger number of them, and presumably with instruments and observational technique at least as advanced as Hipparchus'. Yet each of his own observations of solstices and equinoxes in the *Almagest* is about a day later than the actual event. This consistent and large deviation, and still more the fact that Ptolemy's reported times agree *to the nearest hour* with the times that one would obtain by extrapolating from the observations of Hipparchus and Meton with which they are compared using Hipparchus' tropical year, make it practically certain that the times of the reports are at the least adjusted, if not fabricated, to obtain an appearance of confirmation of Hipparchus' parameter.

p. 133 On Ptolemy's use of 6 sexagesimal places in his mean motion tables Pedersen remarks that "this exactitude is exaggerated" (cf. "illusory exactness," p. 398). In fact it was necessary to *calculate* with this degree of accuracy for mean motions over the 900 odd years from Ptolemy's epoch (747 B.C.) to his own time, in order to maintain accuracy in the second sexagesimal place. It is not necessary to *tabulate* six places, but here, as often, the didactic purpose of the *Almagest* is apparent. [GJT]

p. 134 The reason for the use of 18-year intervals in the mean motion tables has nothing to do with the so-called "Saros," but as Ptolemy himself explains (Hei 1, 209) is to achieve symmetry in the layout of the tables, which were designed for papyrus with a width accommodating 45 lines. See Neugebauer (1975) p. 55 for details. [GJT]

The *Almagest* is not the "only remaining source" for our knowledge of pre-Ptolemaic epicycle/eccenter models. To mention only one other, there is a detailed description of the eccenter and epicycle models for the Sun in Theon of Smyrna's treatise (which Pedersen himself quotes on p. 137), Dupuis (1892) 252ff. [GJT]

p. 140 The word *prosthaphairesis* is attested in two works approximately contemporary with Ptolemy and certainly not influenced by him: the astrologer Vettius Valens (twice in I 6 and once in VIII 7 in Pingree's edition), and the papyrus PSI XV 1491, which is a description of the columns of a table of computed lunar syzygies according to the Babylonian "System B" (see the commentary by Jones in Bartoletti et al. (2008)). The general sense of the word is a numerical quantity that is to be added to or subtracted from another quantity depending on some condition or algorithm. It was likely coined at the time when the algorithms of Babylonian astronomy were first being expressed in Greek texts.

p. 143 There is indeed no "rigorous proof" in the *Almagest* that the true velocity of the Sun equals the mean velocity at a quadrant from the apogee: no justification whatever is offered. But it is visually obvious from the epicycle model, in which the vector of motion on the epicycle points towards the observer in this position. See Neugebauer (1975) p. 57. [GJT]

p. 149 Since Hipparchus did not consider it possible to determine the date of a solstice with a precision of less than $\frac{1}{4}$ day, he probably obtained the season lengths $88\frac{1}{8}$ days from autumnal equinox to winter solstice and $90\frac{1}{8}$ days from winter solstice to vernal equinox by calculation, just as Ptolemy does. Curiously, not a single observation report of the date and time of a winter solstice survives from antiquity.

On Hipparchus' assumption of minimally observable solar parallax see the supplementary note to p. 204.

p. 157 The error resulting from using the known quantity $(\tau - \tau_0)$ instead of the unknown $(t - t_0)$ is utterly negligible. [GJT]

p. 162 The Saros was one of the foundations Babylonian lunar theory. It served as an eclipse period, since after 223 lunar months eclipses of the Moon repeat with similar magnitudes and durations though with a lag of about a third of a

day; circumstances relating to syzygies, such as the time between sunset and moonset on the evening of the new Moon crescent were also predicted by arithmetical manipulations of observations made one Saros in the past. Refinements of the Saros underlie the mathematical lunar algorithms attested in cuneiform tablets from the last three centuries B.C., especially in the so-called System A lunar theory. However, Ptolemy does not necessarily mean Babylonians when he speaks of "ancient mathematicians" who assumed the Saros as a period relation defining the anomalistic and draconitic months in relation to the synodic month; it may already have been known in Greek astronomy in Aristarchus' time, and the relation is the basis of both the eclipse predictions and the display of lunar anomaly in the Hellenistic Antikythera Mechanism (Freeth et al. (2006)). The idea of tripling the Saros to obtain a period approximating a whole number of days appears only in Greek sources (Geminus and the Antikythera Mechanism as well as Ptolemy).

The discrepancy between Ptolemy's value for the mean synodic month, 29;31,50,8,20 days, and the quotient derived from the 4267 month eclipse period was noticed at a very early date, since the "corrected" value 29;31,50,8,9,20 appears in the Arabic translation by al-Ḥajjāj (MS Leiden or. 680 f. 80b), as well as in other Islamic astronomical texts; see the references in Aaboe (1955) to which may be added al-Bīrunī, *al-Qānūn al-Masʿūdī*, see al-Bīrunī (1954–1956) 730, and Elias Bar Shināyā, *Opus Chronologicum*, see Bar Shināyā (1910) 82. [GS]

p. 163 While pointing out that the value 29;31,50,8,20 days does not appear to have come, as Ptolemy says, from the 4267 month period and that this ostensibly Hipparchian parameter, along with Hipparchus' lunar period relations relating anomalistic and draconitic months to synodic months, appear in Babylonian astronomy, Pedersen does not attempt to explain the actual historical processes relating all these parameters, so far as they can be plausibly reconstructed. There is no doubt that Ptolemy has not attempted to give an accurate historical account, his chief concern being to establish how much confidence one should have in Hipparchus' parameters, not where they came from. Ptolemy was almost certainly aware that they were part of the Babylonian heritage, but from his point of view Hipparchus earned the credit for them because he found ways of empirically testing or verifying them. An anonymous commentator on Ptolemy writing in the third century actually attributes the 251 month anomalistic period, which Ptolemy presents as a derivative of the 4267 month period, to a Babylonian authority, Kedenas; see Jones (1990) 20-21.

Kugler (1900)* pp. 20-24 and 40 was the first to point out that the mean synodic month of 29;31,50,8,20 days, the anomalistic period of 251 synodic months, and the latitudinal period of 5458 synodic months are all latent in the Babylonian lunar theory now known as System B. (The specific numbers

as Ptolemy states them are not recorded in any known cuneiform texts, but underlie the algorithms for computing the length of the true synodic month, the Moon's daily motion, and the Moon's latitude.) Kugler maintained that the Babylonians had priority for these parameters, but did not attempt to reconcile their apparent Babylonian origin with Ptolemy's narrative concerning Hipparchus. Aaboe (1955)* argued that, rather than deriving the mean synodic month from the 4267 month eclipse period, as Ptolemy implies he did, Hipparchus actually obtained the 4267 month period as the smallest multiple of the Babylonian 251 month anomalistic period that would bring the Moon from the vicinity of one of its nodes back to the vicinity of a node, so that eclipses could occur at this interval. His purpose was to test the accuracy of the 251 month period relation, since if it was correct, it should be possible to find pairs of eclipses separated by 4267 months, and the time interval between them in days and hours should be constant. Moreover, by multiplying 29;31,50,8,20 days by 4267, Hipparchus obtained a prediction of the length of this supposedly constant interval, and comparison with the observations would thus test the Babylonian mean synodic month. Toomer (1980) clarified the nature of Hipparchus' test, which required (at least) two pairs of observed eclipses separated by 4267 months, such that the Moon was at opposite extreme stages of its anomalistic motion in each pair (i.e. near minimum apparent motion in one pair and maximum in the other) in order that any inequality in the assumed period relations would show up most clearly in the differences between the observed times of the eclipses. The "not too clear" chapter IV 2 (cf. p. 161) is largely concerned with these criteria and the difficulty of satisfying them. Toomer also attempted to identify the specific eclipses that Hipparchus used, in accordance with Ptolemy's statement that the later in each pair was observed by Hipparchus himself and that the earlier one was a Babylonian observation from approximately 345 years earlier, i.e. from the early fifth century B.C.

How the Babylonians obtained these parameters remains obscure, not only because the cuneiform texts contain no explicit remarks about their derivation, but also because it is very difficult to reconstruct plausible routes by which parameters of such high quality could have been obtained from the kind of observational records that we know were available to Babylonian astronomers. (Pedersen's note 3 seriously underestimates the difficulties involved in accurately isolating the lunar periodicities from observations.) Britton (2007) 122-123 suggests that the mean synodic month of 29;31,50,8,20 days was derived after all from the 4267 month eclipse period, but by Babylonian astronomers (rather than Hipparchus) who inferred from observational records that eclipses separated by this number of months are separated by an approximately constant interval of 126007 days plus 15 time degrees (= 1 equinoctial hour). The discrepancy between the quotient ($126007\frac{1}{24}$ days)/4267 and 29;31,50,8,20 can be accounted for as an insignificant adjustment dictated by arithmetical convenience in operating the System B algorithms. If Britton's hypothesis is correct, one may wonder whether Hipparchus learned of the 4267 month period from the Babylonians or rediscovered it.

According to the Babylonian period relation verified by Hipparchus the Moon is supposed to return to the same node precisely after 5458 (mean) synodic months, and so it is possible for two lunar eclipses to be observed separated by this interval. Hipparchus probably had access to such pairs of observations; for example, there was an eclipse observable in Babylon on October 16, 583 B.C., 5458 synodic months before the eclipse of January 27, 141 B.C. that Hipparchus observed in Rhodes (*Almagest* VI 5; Hei 1, 477-478). But to test the accuracy of the period relation, Hipparchus also needed to have eclipses such that the Moon was close to the same stage of its anomalistic motion. A large difference in the Moon's equations at the two dates would result in a time interval between the eclipses that is longer or shorter than 5458 mean synodic months, and would also significantly affect the observed eclipse magnitudes, from which Hipparchus hoped to establish whether the Moon had returned to the same elongation from the node. The situation was thus the converse of his test of the Babylonian period relation for lunar anomaly, where it was necessary to find an interval that was an integer multiple of 251 synodic months but that brought about an approximate return to the same nodal distance. But 5458 synodic months is already more than 441 years, and any multiple would require access to observational records long before the earliest that Hipparchus possessed.

This explains why, according to Ptolemy's later discussion in *Almagest* VI 9 (Hei 1, 525-527), Hipparchus tested the period relation using two observed eclipses that were separated by 7160 synodic months, which is approximately 7770 draconitic months. His procedure seems to have been to find a period that was *almost* equivalent to the Babylonian relation but that was short enough so that there would exist some multiple of it bringing about a near return to the same stage of anomaly while remaining within the chronological range of the available eclipse observations. But there are a number of so far unresolved difficulties in reconciling the details Ptolemy gives of Hipparchus' procedure in IV 2 and VI 9.

p. 174 In his account of Ptolemy's derivation of the size of the Moon's epicycle from three eclipses, Pedersen does not discuss how one determines the position of the observer (T) relative to the three points on the epicycle although this is fundamental to the problem. See Neugebauer (1975) 74–75.

p. 181 The discussion of Ptolemy's method of determining the mean motion of the argument of latitude omits most of the observational conditions (carefully defined by Ptolemy) necessary to make the method valid. In addition to having as long an interval of time as possible between them, the two eclipses must have the Moon near the same node and at the same distance from the Earth, and the observed magnitudes must be equal, with the obscuration in the same direction (either north or south).

In the introduction to this chapter (Hei 1, 326–328), Ptolemy outlines a different method, which he says he had applied at an earlier date but now repudiates because it depended on parameters from Hipparchus that he no

longer trusts. N. T. Hamilton brilliantly realized that this passage is referring to the different parameters of lunar motion in Ptolemy's Canobic Inscription; see Hamilton, Swerdlow, & Toomer (1987). After alluding to other changes he has made in his parameters for the models of Saturn and Mercury, Ptolemy comments that an astronomical theorist should by no means consider it disgraceful to correct his own results or have them corrected by someone else afterwards.

p. 183 Pedersen assumes that the longitudes of both Sun and Moon are observational data. However, the normal procedure for using the astrolabon was to set it to the *computed* position of one body and thus determine the position of the second (as Ptolemy says at V 1, Hei 1, 353, and Pedersen himself implies p. 240 bottom). [GJT] In the present instance Ptolemy does not make it clear whether he set up the astrolabon by aligning the observed Sun to its position computed from Ptolemy's model, or by aligning astrolabon's ecliptic ring so that it exactly shadowed itself.

Pedersen also does not mention that, for the purpose at hand, Ptolemy needs the Moon's *true* longitude (as if seen from the Earth's center), whereas the astrolabon finds the Moon's apparent position affected by parallax. This presents Ptolemy with a methodological problem since he will only be in a position to deduce a theory of parallax after completing the complete deduction of the lunar model and its parameters. In three of the four observations of lunar elongations that he uses in *Almagest* V 3 and 5, he circumvents the difficulty by choosing situations where the longitudinal component of the parallax is negligible. For the fourth, an observation made by Hipparchus (Hei 1, 369), he accepts without comment Hipparchus' own parallax correction, though this would have presumably been based on different mathematical methods and on a value of the Moon's distance from the Earth different from Ptolemy's.

p. 190 The hours of the time difference used by Ptolemy are not $18\frac{1}{3}$ (as one obtains by subtracting the earlier date from the later) but 18; the difference is due to the equation of time. [GJT]

p. 201 Ptolemy's value for the Moon's maximum latitude, 5°, ostensibly comes from observations of the Moon's noon altitude that he describes qualitatively (without specific dates) in V 12 (Hei 1, 407), in situations such that the Moon was simultaneously near the summer solstitial point (Cancer 0°) and near the northern limit of its orbit. These conditions would bring the Moon near enough to the zenith for parallax to be negligible. It is questionable whether Ptolemy actually made such observations; if he did, it would have had to be in A.D. 126, about the beginning of his observing career, or A.D. 145, four years after the last observation reported in the *Almagest* (Newton (1978) 184–186). The 5° maximum was, as Ptolemy says (Hei 1, 388), one of Hipparchus' parameters.

Pedersen fails to give a straightforward account of how one computes the latitude of the Moon from the *Almagest* tables. Instead he gives (6.68) a formula for computing the position of the node which, though correct, is totally unnecessary for latitude computations, and then (p. 205) uses this formula for computing the true argument of latitude by a very roundabout method. All that is necessary is to take the mean argument of latitude directly from the tables and apply the equation to get the true argument of latitude. [GJT]

p. 204 Swerdlow (1969)* 210 showed from Pappus' report of Hipparchus' procedure that Hipparchus considered a parallax of 7 minutes to be "just great enough to be perceptible."

p. 205 The minutes of the time difference are 20, not 50, because Ptolemy has made a correction for the equation of time. [GJT]

p. 207 On the comparison of Ptolemy's value for the mean distance of the Moon with those of earlier astronomers, see Toomer (1974).

p. 208 An important condition for the eclipses used in Ptolemy's analysis is that the Moon must be near the epicycle's apogee at both eclipses.

The reports of the times of the two Babylonian eclipses are, to say the least, misleading. The first, says Pedersen, "began about $16^h\ 45^m$." The facts of Ptolemy's report are: the eclipse began at the end of the 11th hour of night; hence mid-eclipse was about 6 civil hours, or 5 equinoctial hours, after midnight. This corresponds to 5 hours at Alexandria, which is corrected to $4\frac{3}{4}$ hours for the equation of time. Thus $16^h\ 45^m$ is the *middle* of the eclipse counted from *noon* at *Alexandria*. The second eclipse, according to Pedersen, "took place at 11^h." Only by looking at the *Almagest* can we find out that "took place" refers to mid-eclipse, and that "11^h" means "11 equinoctial hours after noon in *Babylon*" (which Ptolemy converts to 9;50 hours after noon at Alexandria). [GJT]

Sosigenes, in the fragment referred to later in Pedersen's footnote, clearly refers to an annular eclipse as an observed event. This Sosigenes was the teacher of Alexander of Aphrodisias and therefore flourished in the later second century A.D.; Pedersen has erroneously conflated him with an earlier man of the same name associated with Julius Caesar. [GJT]

p. 209 Pedersen's account of Hipparchus' treatment of the problem of the distances of the Sun and Moon is inaccurate. His two approaches were contained in a single work, *On Sizes and Distances*, and both were deductions of the lunar distance from an assumed solar distance together with empirical data. In Book 1 of *On Sizes and Distances*, he assumed that the Sun's distance was infinite, whereas in Book 2 he assumed that the Sun's distance was 490 Earth-radii. For details see Toomer (1974).

p. 212 It seems implausible that Ptolemy would have gone out of his way to preserve Aristarchus' ratio of solar to lunar distance, not only because of the crudeness of Aristarchus' data but also because Aristarchus determines the ratio presupposing a configuration in which the Moon is near quadrature, which is precisely where Ptolemy's model brings the lunar epicycle closest to the Earth. See Carman (2009) 207, note 5. Carman argues that Ptolemy probably already had thought of the hypothesis of nesting spheres that he presented in the *Planetary Hypotheses*, and that the ratio of distances in the *Almagest* was determined by the space required between the solar and lunar models for those of Mercury and Venus. The chief difficulty of this is that it becomes hard to explain why Ptolemy says nothing about the possibility of nested spheres in IX 1.

p. 219 Ptolemy's "improved" method of calculating longitudinal and latitudinal parallax involves more than a "slight" approximation, since, as Pedersen himself remarks, it treats a spherical triangle as plane; he does not remark that this triangle is very large, so that the error can be considerable. On the whole question see Neugebauer (1975) 116–117. [GJT]

p. 222 Ptolemy explicitly states (Hei 1, 464) that his tables use 25-year intervals because 25 Egyptian years are about equal to 309 synodic months. [GJT]

p. 231 Ptolemy's proof that solar eclipses cannot occur at one-month intervals contains the important qualification "in the *oikoumene*." He was undoubtedly aware that solar eclipses can occur at one-month interval in opposite hemispheres, since Hipparchus is known to have shown this; see Neugebauer (1975) 322. [GJT]

p. 244 There are serious errors in Pedersen's computations of the parallax here. Unfortunately, he gives only the results of his computations, so the sources of the errors are not clear. But the figure of 77° for the angle between the ecliptic and the vertical at the first observation is impossible (it is certainly considerably more than 90°). This changes the longitudinal component of parallax from negative to positive, and thus Manitius is correct in stating that the longitudinal parallax was positive in the first observation and negative in the second (contrary to Pedersen's denial at p. 245 note 4). The situation is further complicated by Pedersen's forgetting that Ptolemy's parallax formulas, which he uses for the calculations on p. 244, are designed for going from true to apparent positions, whereas in the equation he develops on pp. 242-243 (8.3 and 8.6) the parallax is applied to the apparent position to get the true. Therefore in those equations the opposite sign must be applied to π_1 and π_2, and equation (8.7) should read
$\pi_1 - \pi_2 = 0;5°$. [GJT]

p. 248. One of the principal inspirations of the Islamic theories of trepidation, including that of *On the Motion of the Eighth Sphere* (Thābit's authorship of

which is now regarded is doubtful), is a passage in Theon of Alexandria's shorter commentary on the *Handy Tables* that describes a belief of certain astrologers that the equinoctial and solstitial points have a slow oscillating motion. See Ragep (1996) and Jones (2010).

p. 251 The identifications of Peters and Knobel are now superseded by those in Toomer (1984) and Graßhoff (1990). There still remain numerous uncertainties in the identification of individual stars.

p. 253 The systematic error in the tropical longitudes of the stars in Ptolemy's catalogue is simply the error of approximately 1° in the positions of the tropical and equinoctial points arising from the errors in his alleged solstice and equinox observations (see p. 132 with its supplementary note). Ptolemy's lunar theory is determined by observations of the Moon relative to the Sun, and his stellar longitudes are ostensibly determined by observations relative to the Moon, so that the error in the tropical frame of reference inevitably carries through to the catalogue (and from it subsequently to the theories of the planets).

p. 254 Nallino (Battani (1899–1907)* v. 2, 269–70) showed that the conclusions of Björnbo (1901)* were groundless, being based on a misunderstanding of a passage in al-Battanī. Islamic astronomers had exactly the same information about Menelaus' astronomical activities as we have, namely the report (in the *Almagest*) of *two* observations by him of occultations of fixed stars by the Moon. [GJT]

p. 255 Graßhoff (1990) 99–121 has exposed grave defects in the arguments of Vogt (1925), and demonstrates that a proper analysis of Vogt's own data shows a high degree of statistical dependence between the coordinates in Ptolemy's catalogue and the positional information in Hipparchus' Aratus commentary. See also Duke (2002a). Duke (2002b) has shown that Hipparchus regularly employed equatorial, not ecliptic, coordinates for fixed star positions. An enormous scholarly literature has arisen on Ptolemy's catalogue and its possible dependence on earlier catalogues by Hipparchus or others, some of which is reviewed in Swerdlow (1992).

p. 256 Pedersen's argument that Hipparchus must have been a better observer of stellar longitudes than Ptolemy is fallacious. $\Delta\lambda_2$ incorporates the roughly 1° systematic error of all Ptolemy's longitudes, which has nothing to do with inaccuracies in Ptolemy's observations of fixed stars, but, as remarked in the supplementary note to p. 253, stems from his faulty equinoxes and solstices. When one considers, not the size of the error, but the amount of fluctuation about the mean error, quite a different picture appears. [GJT] However, one also should not conclude from such data that Hipparchus was a *worse* observer than Ptolemy, since the positional data in the Aratus commentary from which Vogt reconstructed the Hipparchian longitudes, which are only

expressed to a precision of single or half degrees, could well be roundings of more precise coordinates.

p. 259 It is by no means "established that the classification of... stars... in 6 orders of magnitude goes back at least to Hipparchus." This may be true, but there is no explicit evidence that Hipparchus used such a classification. The earliest occurrence is in Manilius (early first century A.D.), Book V, lines 710–717. [GJT]

p. 261 Although the great eccentricity of Mercury's orbit is undoubtedly one reason for the complexity of Ptolemy's model for this planet, a greater reason is Mercury's proximity to the Sun, as a result of which it is observable to the naked eye for comparatively brief intervals of its synodic cycles, and observable together with stars (so that its position can be determined) even less often.

p. 269 The period relations for the planets that Ptolemy ascribes to Hipparchus, namely

Saturn:	$57\ T_a = 2\ T_t = 59^a$
Jupiter:	$65\ T_a = 6\ T_t = 71^a$
Mars:	$37\ T_a = 42\ T_t = 79^a$
Venus:	$5\ T_a = 8\ T_t = 8^a$
Mercury:	$65\ T_a = 6\ T_t = 71^a$

were also known to Babylonian astronomers, where they are the basis of the variety of observational record known as Goal Year Texts.

p. 278 A simple eccenter-and-epicycle model is increasingly inaccurate as a representation of a planet's motion as the eccentricity of the planet's actual orbit around the Sun is increased, and the discrepancy is most pronounced when the planet is moving nearest to the Earth, around opposition for an outer planet and around inferior conjunction for an inner planet. One way that it shows up is in the sizes, durations, and spacing of the planet's retrogradations, which disagree with what one would predict from any simple model; Ptolemy alludes to this, though in not entirely clear terms, in X 6 (Hei 2, 316). It is very likely that the inadequacy of the simple model first became apparent (either to Ptolemy or to an unknown predecessor) in the case of Mars, since this was the planet with the most eccentric orbit that could be accurately observed around the time of its retrogradation. For diverse reconstructions of how the need for an equant and its optimal placement might have been discovered, see Evans (1984), Swerdlow (2004), Jones (2004), and Duke (2005).

p. 283 In fact if one repeats Ptolemy's iterative procedure for determining the eccentricities of each of the three outer planets with accurate calculations at every step, one finds that it takes one more iteration than Ptolemy gives; the reason is that he has deliberately introduced arithmetical inaccuracies that accelerate the convergence. See Duke (2005c).

p. 288 In this section Pedersen misrepresents what Ptolemy is doing in IX 7 (and the corresponding chapters for the other planets). According to him, this is a "test" of the correctness of the mean motions employed in the construction of the models for the planets. But Ptolemy never calls this procedure a "test": he describes it, properly, as a method of "correcting the mean motions." The sequence was as follows. Ptolemy used mean motions derived from the Babylonan period relations transmitted by Hipparchus (e.g. Saturn has 57 returns in anomaly and 2 returns in longitude in 59 years) to establish the parameters used in constructing the models of the planets. He then used these models, combined with observations of longitudes as far apart as possible, to correct the mean motions, as described e.g. in XI 7. But in his final presentation in the *Almagest*, when detailing the mean motions in IX 3, he anticipates his later corrections (exactly as he had done for the lunar model). He makes no secret of this (Hei 2, 213). [GJT]

p. 292 Columns III and IV are combined into a single column for the equation of center in medieval astronomical tables only because their format is derived from Ptolemy's own *Handy Tables*, which already adopt this improvement. [GJT]

p. 298 Swerdlow (1989) gives an interesting analysis of the *Almagest*'s deduction of the models of Venus and Mercury, showing that it is an artificial, didactic approach that cannot represent the way that Ptolemy originally arrived at his parameters.

p. 309 On the doubtful identification of Ptolemy's acquaintance Theon with Theon of Smyrna see the supplementary note to p. 13. The third century B.C. observations dated according to the calendar "according to Dionysius" may have been made by Dionysius himself (concerning whom we have no information except that he worked at Alexandria) but are very unlikely to be the work of Timocharis, whose attributed observations use either the Egyptian or Callippic calendars. On the sources and character of Ptolemy's planetary observation reports from the third century see Jones (2006).

p. 313 Pedersen gives no explanation why he defines the eccentricity *e* as *half* the distance TF (contrast the procedure for Venus, Fig. 10.2 on p. 302, where *e* is the *whole* distance TD). The reason is the peculiar model for Mercury (explained p. 315ff). [GJT]

p. 355 Pedersen's paraphrase misleadingly suggests that Ptolemy invokes a personal god as the arbiter of simplicity; in fact Ptolemy only speaks of the heavenly bodies themselves as "divine." [GJT]

p. 361 In fig. 12.4 the mean motion is represented as taking place about the center of the "wheel" causing the deviation in latitude, and there is nothing in the accompanying explanation to make one think otherwise. But Ptolemy devotes a long passage (Hei 2, 531-532) to explaining why its mean motion must take place about an eccentric point. [GJT]

p. 365 Consideration of Ptolemy's latitude models from a heliocentric perspective is a more complex problem than Pedersen's discussion suggests. See now Swerdlow (2005).

p. 367 For all planets except Mercury (which is not in question here), finding the mean centrum from the true is a geometrical problem just as simple as finding the true centrum from the mean. [GJT]

p. 369 Given the peculiarity of the latitude theory for the inner planets, in which the planes of the deferents move up and down so that different points are northermost ("at the top") at different times, we would be well advised to avoid the term "top of the deferent" altogether in this context, as Ptolemy does. However, *if* one uses it, then one ought to say that for Mercury the top of the deferent is 180° from the apogee. Similarly the equation (12.31) ought to read, for Mercury,

$$\lambda_d = c - 90° .$$

However, λ_d means "the longitude of the ascending node," and it is confusing to talk about "the ascending node" here, when the epicycle of one planet (Mercury) never ascends above the plane of the ecliptic, and that of the other never descends below it. Hence Manitius' statement, which Pedersen (p. 376 note 7) finds "curious," that Mercury has no ascending node; and hence Ptolemy's avoidance of the term for Venus and Mercury; he always refers to "the node in the additive [or subtractive] semi-circle."

p. 381 The error involved in Ptolemy's approximative formula for the "third latitude" is actually small, e.g. for Venus the maximum error is less than 7 minutes. [GJT]

p. 388 In fact Ptolemy gives a long and explicit derivation of the "arcus visionis" from observed values of elongation from the Sun at a latitude with a longest day of $14\frac{1}{4}$ hours. For the method see Neugebauer (1975) 235. [GJT]

p. 392 Some of the mean motions in the *Planetary Hypotheses* are slightly different from their *Almagest* counterparts; for a discussion, see Duke (2009).
The *Planetary Hypotheses* models for planetary latitude are compared with those of the *Almagest* and *Handy Tables* in Swerdlow (2005).

p. 396 For a lucid review of the solid models of *Planetary Hypotheses* II see Murschel (1995).

p. 397. The tables that Ptolemy promises to provide at the end of *Planetary Hypotheses* II were not the *Handy Tables* (which Ptolemy produced between the *Almagest* and *Planetary Hypotheses*) but a similarly formatted set of tables exclusively of mean motions of the heavenly bodies.

Tihon (1985) has shown that there is no foundation for the formerly widely held view that Theon of Alexandria redacted the *Handy Tables*. The versions extant in medieval manuscripts undoubtedly contain much extraneous material, but the core of essential tables appears to substantially preserve what Ptolemy published. A critical edition by Anne Tihon and Raymond Mercier is in progress.

p. 401 Pedersen's account of the early history of Greek astrology reflects the highly speculative scholarship of the late nineteenth and early twentieth century, and the role he ascribes to Posidonius in particular can be discounted. For a careful sifting of the evidence concerning when, where, and how Greek astrology came into being see Pingree (1997) 21–29.

p. 403 The Arabic version of the *Planispherium* has now been published with an English translation in Sidoli & Berggren (2007).

p. 406 For recent publications of the *Geography* see the supplementary note to p. 109, and on the Canobic Inscription see the supplementary note to p. 12.

p. 408 Observation 2: this eclipse was also used by Hipparchus (*Almagest* VI 9, Hei 1, 526). [GJT]

p. 412 What Ptolemy calls the "Chaldaeic" era (in observations 26 and 28-29) is what is now called the "Seleucid era," which was the era employed in the original Babylonian observation records on cuneiform tablets. [GJT]

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